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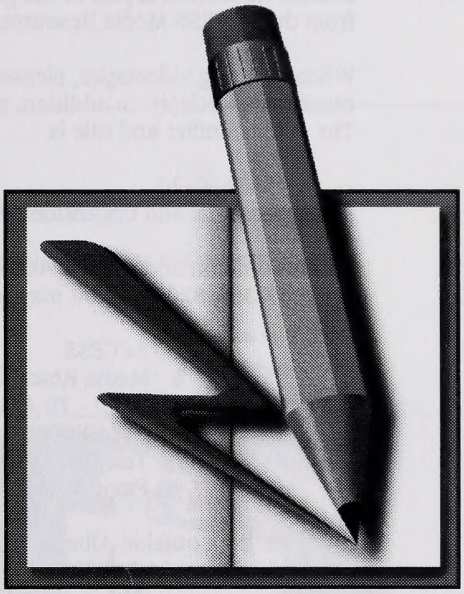
MATH FACTOR

EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Lesson 1: Sketching and Graphing Exponential Functions

Student Workbook

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This guide accompanies **Sketching and Graphing Exponential Functions** from the Math Factor series.

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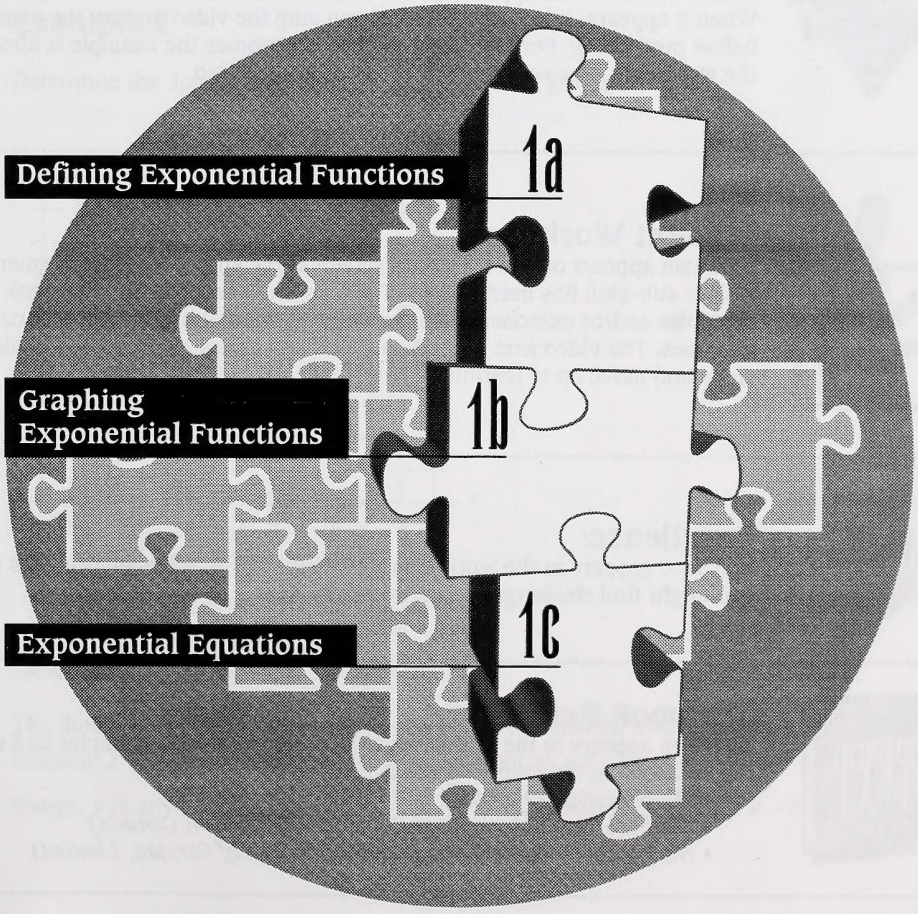
Lesson 1: SKETCHING AND GRAPHING EXPONENTIAL FUNCTIONS

LEARNER EXPECTATIONS

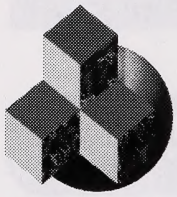
By the end of Lesson 1 you should be able to

- sketch the graph of $y = a^x$, $a > 0$
- use graphs to estimate values of roots and powers
- draw graphs and analyze them using calculators or computers
- determine domain and range
- solve and verify exponential equations

This lesson has three parts:



EXPLANATION OF ICONS



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Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

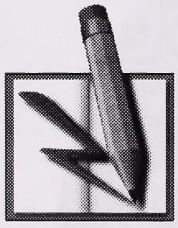
Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.



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This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

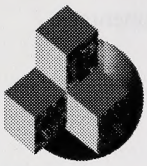
Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*



Building Blocks

Do you remember what the domain and range of a function are and how to work with exponents?

- The domain of a function is the set of values that the independent variable may take on.
- Similarly, the range is the set of variables that the function takes on.

Example 1

Determine the domain and range for $f(x)=\{(3,2)\ (4,8)\ (-9,7)\}$.

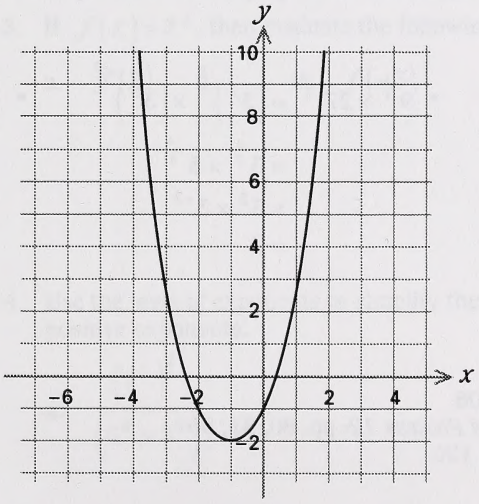
Solution

Domain: $\{3,\ 4,\ -9\}$

Range: $\{2,\ 8,\ 7\}$

Example 2

Determine the domain and range for the following graph.



Solution

The domain and range of a function can only be estimated from a graph.

Domain: x is a member of real numbers, denoted as $\{x \mid x \in R\}$.

Range: y is any value greater than or equal to -2, denoted as $\{y \mid y \geq -2,\ y \in R\}$.

The **laws of exponents** govern operations on expressions that contain exponents.

Product Law: $a^m \times a^n = a^{m+n}$

Quotient Law: $\frac{a^m}{a^n} = a^{m-n}$

Power of a Power Law: $\left(a^m\right)^n = a^{mn}$

Power of a Product Law: $(ab)^m = a^m b^m$

Power of a Quotient Law: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Note: m and n are real numbers.

Also, recall three definitions relating to powers:

- $a^0 = 1$
- $a^{-n} = \frac{1}{a^n}$; $\frac{1}{a^{-n}} = a^n$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{a^m}$

These laws and definitions can be used to simplify or evaluate expressions. Study the following examples.

- $5^4 \times \frac{5^3}{\left(5^3\right)^2} = \frac{5^{4+3}}{5^{3\times 2}}$
  $= \frac{5^7}{5^6}$
  $= 5^{7-6}$
  $= 5$
- $9^{\frac{3}{2}} \times 27^{\frac{-2}{3}} = \left(3^2\right)^{\frac{3}{2}} \times \left(3^3\right)^{\frac{-2}{3}}$
  $= 3^{\frac{6}{2}} \times 3^{\frac{-6}{3}}$
  $= 3^3 \times 3^{-2}$
  $= 3$
- $\left(\frac{a^3 b^2}{b^{-2}}\right)^4 = \frac{a^{12} b^8}{b^{-8}}$
  $= a^{12} b^{8+8}$
  $= a^{12} b^{16}$

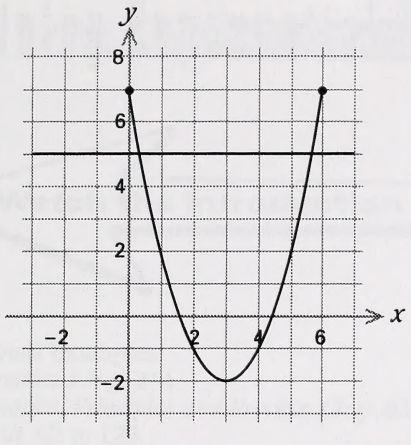
Additional Examples
Mathematics 12: pp. 304-306
Mathematics: Principles and Process 12: pp. 80, 82, 86
Holtmath 12: pp. 116, 118, 120

Exercises

1. Determine or approximate the domain and the range of the following.

   a. $f(x) = \{(2,4), (-3,6), (5,9)\}$

b.



2. Find the value of each of the following:

a. 4^0 b. 7^{-2} c. -3^4

3. If $f(x) = 2^x$, then evaluate the following:

a. $f(2)$ b. $f(-3)$ c. $f(-1.5)$

4. Use the laws of exponents to simplify the following expressions. Express with positive exponents.

a. $\dfrac{\left(3^2\right)^2}{\left(3^3 \times 3^2\right)}$

b. $\dfrac{\left(t^{x-3}\right)\left(t^{2x+5}\right)}{t^2}$

c. $3^{\frac{1}{7}x} \times \dfrac{27^{\frac{2}{7}x}}{9^{2x-1}}$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 307-309 (Questions 1, 2, 6, 12)
Mathematics: Principles and Process 12: p. 81 (Questions 1-5, 6, 8)
Holtmath 12: p. 117 (Questions 4-6, 11), p. 119 (Question 5)

Defining Exponential Functions

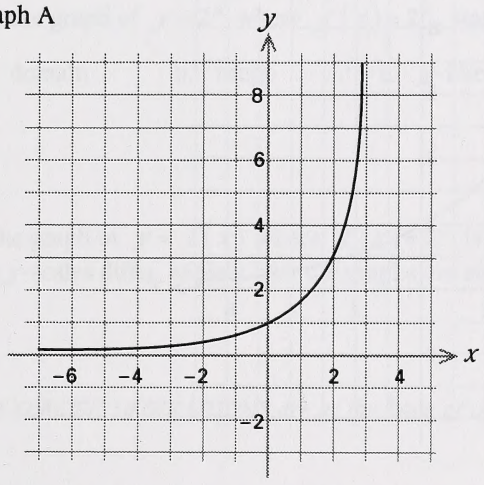
Watch the Introduction and Video Segment 1a.

Additional Examples
Mathematics 12: p. 314
Mathematics: Principles and Process 12: p. 93
Holtmath 12: p. 125

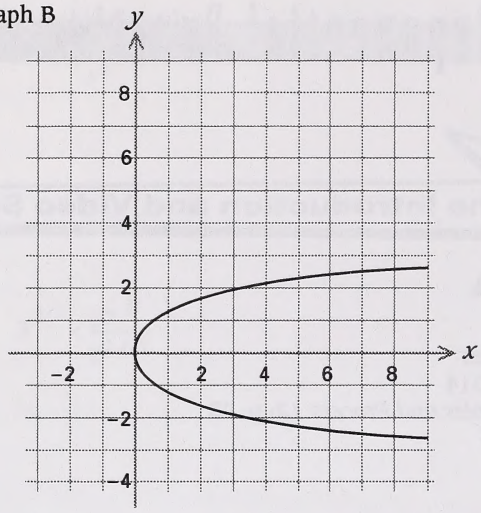
Exercises

1. Which of the following graphs are exponential?

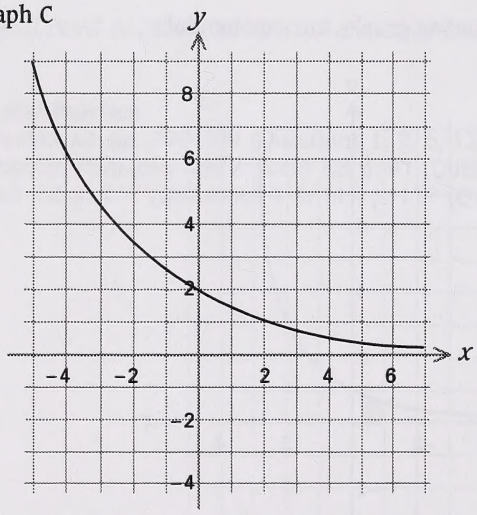
Graph A



Graph B



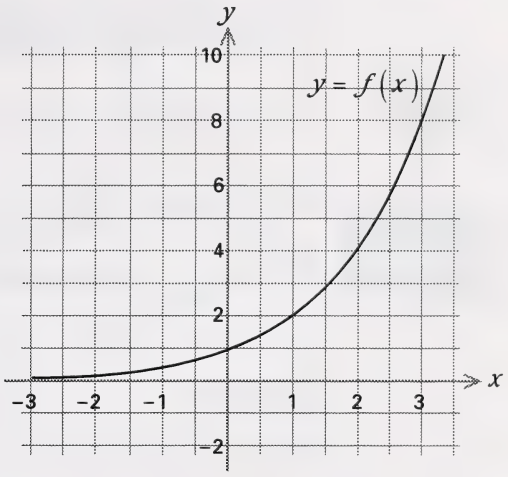
Graph C



2. a. In question 1, which graph shows exponential growth?

b. Which graph shows exponential decrease?

3. The graph of $y = f(x)$, where $f(x) = 2^x$, is given.
Use the graph to estimate the value of the following (to the nearest tenth):



a. $f(x) = 2^{2.5}$

b. x, if $2^x = 3$

c. x, if $2^x = 45$

d. $f(x) = 2^{-0.5}$

4. For the graph of $y = 2^x$ where $f(x) = 2^x$, state the following:

a. domain b. range c. y-intercept d. x-intercept

5. If the graph of $y = f(x)$ where $f(x) = 2^x$ is drawn on two different grids where the y-scales differ, explain how the graphs are similar and how they are different.

EXERCISE 1a SOLUTIONS are at the back of this workbook.

Graphing Exponential Functions

Challenge yourself

Show, algebraically, that the graphs of $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$ would be the same.

Watch Video Segment 1b.

Example

Sketch the graphs of $y = f(x)$, where $f(x) = 3^x$, and $y = g(x)$, where $g(x) = \left(\frac{1}{3}\right)^x$; then state the domain, range, y-intercept and x-intercept.

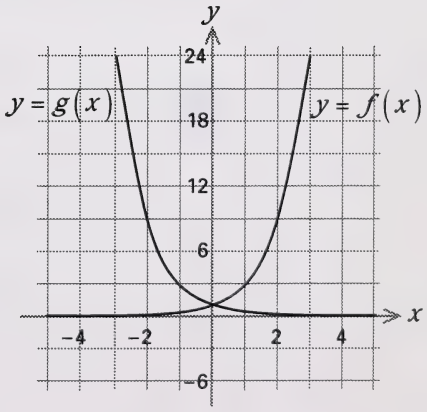
Solution

Step 1: Construct a table of values for $f(x)$ and $g(x)$.

x	-3	-2	-1	0	1	2
$f(x)$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	-3	-2	-1	0	1	2
$g(x)$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

Step 2: Plot the points on a coordinate system and join them using a continuous curve (as shown in the graph on the right).



Therefore, for both the graphs of $y = f(x)$ and $y = g(x)$ the domain, range, y-intercept and x-intercept are as follows:

Domain: x is a member of the real numbers.

$$\{x \mid x \in R\}$$

Range: y is greater than zero.

$$\{y \mid y > 0\}$$

The y-intercept is 1.
There is no x-intercept.

Note: The graph of $y = f(x)$ is an exponential increase curve; as x increases, y increases. Exponential increase is known as exponential growth.
The graph of $y = g(x)$ is an exponential decrease curve; as x increases, y decreases. Exponential decrease is known as exponential decay.
The graphs of $y = f(x)$ and $y = g(x)$ are symmetrical about the y-axis.

Additional Examples
Mathematics 12: pp. 314-316
Mathematics: Principles and Process 12: p. 93
Holtmath 12: p. 126 (Example 3)

Exercises

6. Complete the following tables of values and sketch the graphs on the grid provided on the next page.

a. $y = f(x)$, where $f(x) = 10^x$

x	$f(x)$
−2	
−1	
0	
1	
1.5	

b. $y = g(x)$, where $g(x) = \left(\frac{1}{10}\right)^x$

x	$g(x)$
−1.5	
−1	
0	
1	
2	

c. $y = h(x)$, where $h(x) = 5^x$

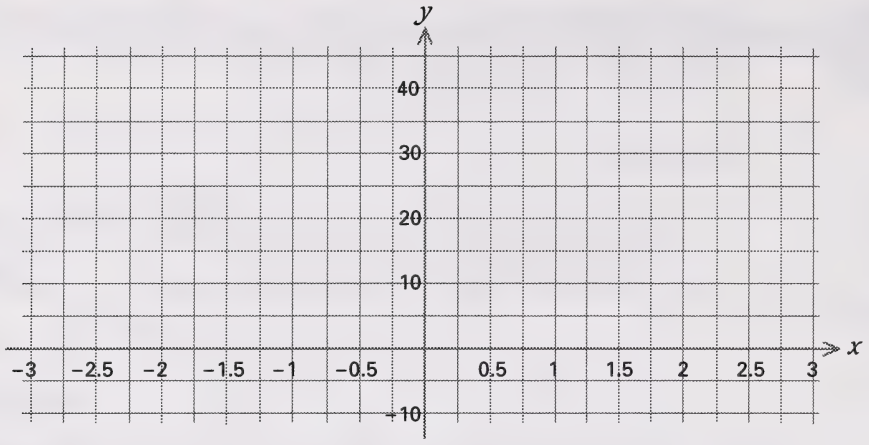
x	$h(x)$
−2	
−1	
0	
1	
2	

d. $y = k(x)$, where $k(x) = \left(\frac{1}{5}\right)^x$

x	$k(x)$
−2	
−1	
0	
1	
2	

e. $y = m(x)$, where $m(x) = 1^x$

x	$m(x)$
−2	
−1	
0	
1	
2	



7. Use the graphs in question 6 to answer the following:

   a. State the x-intercepts and y-intercepts of each function.

   b. State the domain and range of each function.

   c. State the intersection point of the graphs of the functions.

   d. Explain when the graphs of exponential functions will go through this point.

8. Specifically describe the following graphs.

   a. The graph of $y = f(x)$, where $f(x) = a^x$, when $a > 1$.

   b. The graph of $y = f(x)$, where $f(x) = a^x$, when $0 < a < 1$.

   c. The graph of $y = f(x)$, where $f(x) = a^x$, when $a = 1$.

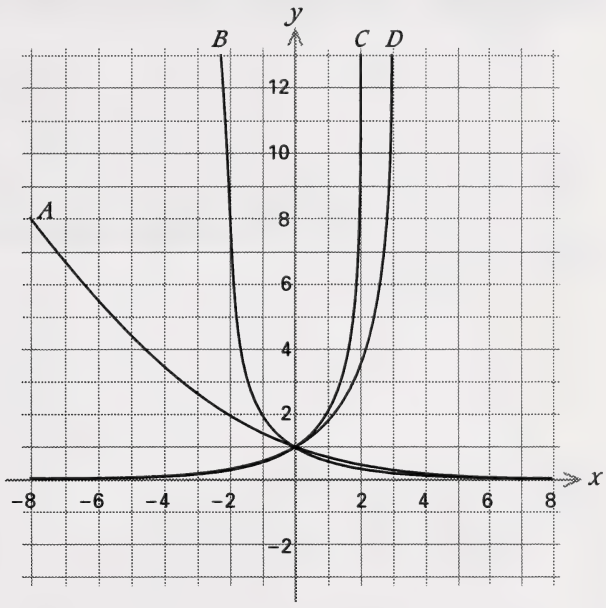
9. Which of the graphs A, B, C and D correspond to the following equations?

a. $y = f(x)$, where
$f(x) = 4^x$

b. $y = g(x)$, where
$g(x) = \left(\frac{9}{4}\right)^x$

c. $y = h(x)$, where
$h(x) = \left(\frac{3}{4}\right)^x$

d. $y = k(x)$, where
$k(x) = \left(\frac{1}{3}\right)^x$

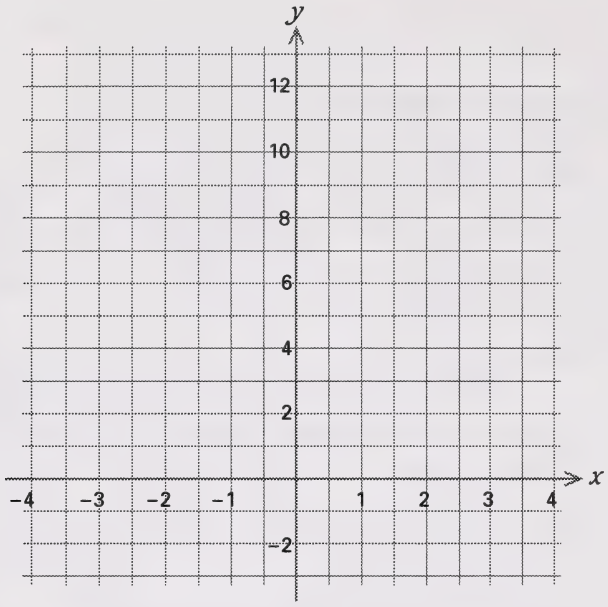


10. a. Draw the graph of $y = 3(2)^x$ on a coordinate system.

b. How does it differ from the graph $y = 2^x$?

EXCELLENCE

11. a. Sketch the graphs of $y = f(x)$ where $f(x) = \frac{1}{2}(2)^x$ and $y = g(x)$ where $g(x) = 2^{x-1}$.



b. Show, algebraically, why the functions $f(x)$ and $g(x)$ are equivalent.

EXERCISE 1b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 317-318 (Questions 3-5)
Mathematics: Principles and Process 12: p. 95 (Questions 1, 3, 4)
Holtmath 12: p. 127 (Questions 4, 11)

Exponential Equations

Challenge yourself

Find the value of x if $16^{x-2} = \left(8^{x-1}\right)(4)$.

Watch Video Segment 1c.

Example 1

Solve for x in the equation $3^x = \left(\frac{1}{27}\right)^{\frac{3}{2}}$.

Solution

$3^x = \left(\frac{1}{27}\right)^{\frac{3}{2}}$

$3^x = \left(27^{-1}\right)^{\frac{3}{2}}$ $\left(\frac{1}{a^n} = a^{-n}\right)$

$3^x = (27)^{-\frac{3}{2}}$ Power of a Product Law

$3^x = \left(3^3\right)^{-\frac{3}{2}}$ 27 expressed as a power of 3

$3^x = \left(3^{-\frac{9}{2}}\right)$ Power of a Product Law

$\therefore x = -\frac{9}{2}$ If $x^m = x^n$, then $m = n$.

Always check or verify that the answer is correct by substituting the answer into the original equation.

L.S.	R.S.
$3^x = 3^{-\frac{9}{2}}$	$\left(\frac{1}{27}\right)^{\frac{3}{2}} = 7.128$
$= 7.128$	

L.S. = R.S.

Since the left side of the equation is equal to the right side, the solution is correct.

Example 2

Solve for x in the equation $125\left(25^{1-x}\right)=1$.

Solution

$125\left(25^{1-x}\right)=1$

$5^3\left[\left(5^2\right)^{1-x}\right]=1$ — 125 and 25 expressed as powers of 5

$5^3\left(5^{2(1-x)}\right)=1$

$5^3\left(5^{2-2x}\right)=1$ — Power of a Product Law

$5^{3+2-2x}=1$ — Product Law

$5^{5-2x}=5^0$ — $5^0 = 1$, since $a^0 = 1$

$\therefore 5-2x=0$ — If $x^m = x^n$, then $m = n$.

$5=2x$

$x=\frac{5}{2}$ or 2.5

Check the answer by substituting it into the original equation.

L.S.	R.S.
$125\left(25^{1-\frac{5}{2}}\right)$	1
$=125\left(25^{-\frac{3}{2}}\right)$	
$=125\left(\frac{1}{125}\right)$	
$=1$	

L.S. = R.S.

Thus, the solution is correct.

You are probably asking what happens if the bases cannot be the same. This question will be answered in **Lesson 4**.

Additional Examples
Mathematics: Principles and Process 12: pp. 89, 90
Holtmath 12: p. 122 (Example 2)

Exercises

12. Solve the following exponential equations.

a. $7^{x+1} = 343$

b. $x^{\frac{1}{2}} = 7$

c. $2^{3x} = \frac{1}{2}$

d. $2\left(5^{2x-9}\right) = 250$

e. $27^{3+x} = \frac{1}{81}$

f. $x^{-\frac{2}{3}} = 64$

g. $\left(\frac{1}{9}\right)^{1.5x} = \left(\frac{1}{27^x}\right)^{x-1}$

h. $\frac{1}{8} = \left(2^x\right)^{x-4}$

EXERCISE 1c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics: Principles and Process 12: pp. 90, 91 (Questions 4, 5)
Holtmath 12: p. 123 (Questions 1, 2)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Exponents, the exponential function and exponential equations are all concepts that you should master before trying the remaining lessons in this module. You should be skillful with the exponential function and solving exponential equations before you continue on to **Lesson 2: Exponential Growth and Decay**.

Solutions

BUILDING BLOCKS

1. a. Domain: $\{2, -3, 5\}$ Range: $\{4, 6, 9\}$
   b. Domain: $\{0 \le x \le 6\}$ Range: $\{-2 \le y \le 7\}$

2. a. $4^0 = 1$
   b. $7^{-2} = \left(\frac{1}{49}\right)$
   c. $-3^4 = -81$

3. a. $f(2) = 2^2 = 4$
   b. $f(-3) = 2^{-3} = \frac{1}{8}$
   c. $f(-1.5) = 2^{-1.5} = 0.354$

4. a. $\frac{\left(3^2\right)^2}{\left(3^3 \times 3^2\right)} = \frac{3^4}{3^5} = \frac{1}{3}$
   b. $\frac{\left(t^{x-3}\right)\left(t^{2x+5}\right)}{t^2} = t^{x-3+2x+5-2} = t^{3x}$
   c. $3^{\frac{1}{7}x} \times \frac{27^{\frac{2}{7}x}}{9^{2x-1}} = 3^{\frac{1}{7}x} \times \frac{\left(3^3\right)^{\frac{2}{7}x}}{\left(3^2\right)^{2x-1}}$
   $= \frac{3^{\frac{1}{7}x} \times 3^{\frac{6}{7}x}}{3^{4x-2}}$
   $= \frac{3^x}{3^{4x-2}}$
   $= 3^{x-4x+2}$
   $= 3^{2-3x}, x > \frac{2}{3}$

EXERCISE 1a

1. Graphs A and C are exponential.

2. a. Graph A
   b. Graph C

3. a. 5.7 b. 1.6 c. 2.2 d. 0.7

4. a. Domain: $\{x \mid x \in R\}$
   b. Range: $\{y \mid y > 0\}$
   c. The y-intercept is 1.
   d. There is no x-intercept.

5. The graphs will have the same shape (an exponential increase curve), but the graph with a larger scale on the y-axis will not be as steep and will not increase as rapidly as the other graph.

EXERCISE 1b

6. $f(x) = 10^x$

x	$f(x)$
−2	0.01
−1	0.1
0	1
1	10
1.5	31.6

$g(x) = \left(\frac{1}{10}\right)^x$

x	$g(x)$
−1.5	31.6
−1	10
0	1
1	0.1
2	0.01

$h(x) = 5^x$

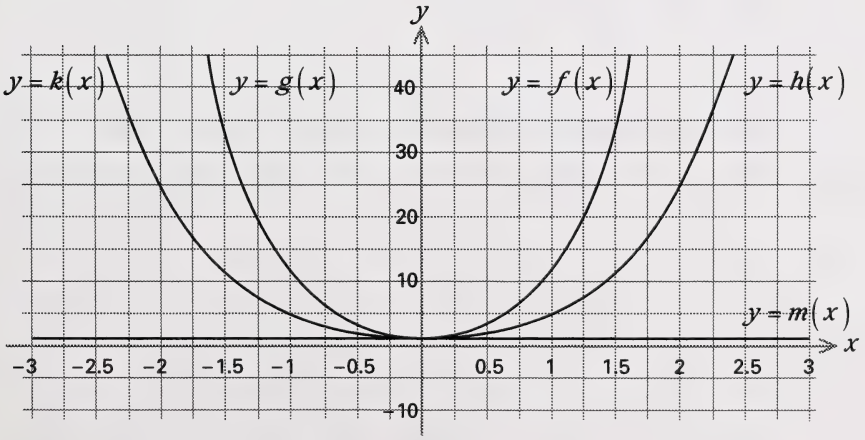
x	$h(x)$
−2	0.04
−1	0.2
0	1
1	5
2	25

$k(x) = \left(\frac{1}{5}\right)^x$

x	$k(x)$
−2	25
−1	5
0	1
1	0.2
2	0.04

$m(x) = 1^x$

x	$m(x)$
−2	1
−1	1
0	1
1	1
2	1



7. a. For all the functions, there are no x-intercepts, and the y-intercept is 1.

   b. For all the functions, the domain is $\{x \mid x \in R\}$. The range for $m(x)$ is $\{y \mid y = 1\}$.

   The range for all the other functions is $\{y \mid y > 0\}$.

   c. The intersection point is $(0, 1)$.

   d. The graphs will go through this point when $c = 1$ in the function $y = cf(x)$.

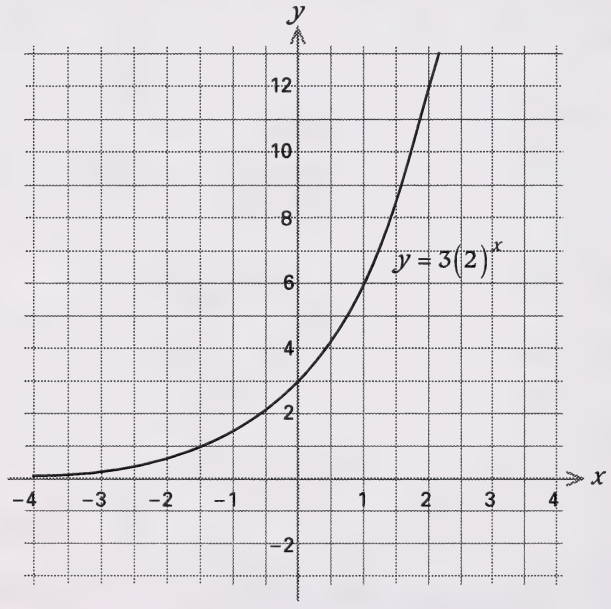
8. a. This graph is an exponential increase curve. As a increases, the graph increases more rapidly (i.e., is steeper).

   b. If $0 < a < 1$, then the graph is an exponential decrease curve (as x increases, y decreases). As a approaches 0, the graph decreases more rapidly. As x approaches 1, the graph is flatter.

   c. If $a = 1$, the graph becomes $y = 1$ (a horizontal line with y–intercept 1).

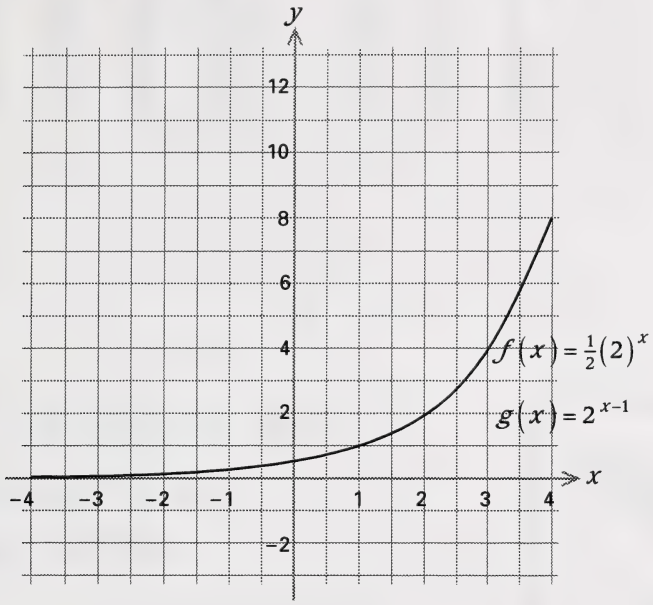
9. a. *C* b. *D* c. *A* d. *B*

10. a.



   b. The graph of $y = 3(2)^x$ has been stretched by a factor of 3; that is, if $f(x) = 2^x$ and $g(x) = 3(2)^x$, then $g(x) = 3f(x)$. (The y-values of $y = f(x)$ have been multiplied by 3.)

11. a.



b. $f(x)=g(x)$

$$\frac{1}{2}(2)^x = 2^{x-1}$$

$$2^{-1}\times 2^x = 2^{x-1}$$

$$2^{x-1} = 2^{x-1}$$

EXERCISE 1c

12. a. $7^{x+1}=343$

$7^{x+1}=7^3$

$\therefore x+1=3$

$x=2$

b. $x^{\frac{1}{2}}=7$

$\left(x^{\frac{1}{2}}\right)^2=7^2$

$x=49$

c. $2^{3x}=\frac{1}{2}$

$2^{3x}=2^{-1}$

$\therefore 3x=-1$

$x=-\frac{1}{3}$

d. $2\left(5^{2x-9}\right)=250$

$5^{2x-9}=125$

$5^{2x-9}=5^3$

$\therefore 2x-9=3$

$x=6$

e. $\left(27^{3+x}\right)=\frac{1}{81}$

$\left(3^3\right)^{3+x}=\frac{1}{\left(3^4\right)}$

$3^{9+3x}=3^{-4}$

$\therefore 9+3x=-4$

$3x=-13$

$x=-\frac{13}{3}$

f. $x^{-\frac{2}{3}}=64$

$\left(x^{-\frac{2}{3}}\right)^{-\frac{3}{2}}=64^{-\frac{3}{2}}$

$x=\left(2^6\right)^{-\frac{3}{2}}$

$x=2^{-9}$

$x=\frac{1}{512}$

g. $\left(\frac{1}{9}\right)^{1.5x}=\left(\frac{1}{27^x}\right)^{x-1}$

$\left(3^{-2}\right)^{1.5x}=\left(3^{-3x}\right)^{x-1}$

$3^{-3x}=3^{-3x^2+3x}$

$\therefore -3x=-3x^2+3x$

$3x^2-6x=0$

$3x(x-2)=0$

$\therefore x=0$ or $x=2$

h. $\frac{1}{8}=\left(2^x\right)^{x-4}$

$2^{-3}=2^{x^2-4x}$

$\therefore -3=x^2-4x$

$0=x^2-4x+3$

$0=(x-1)(x-3)$

$\therefore x=1$ or $x=3$

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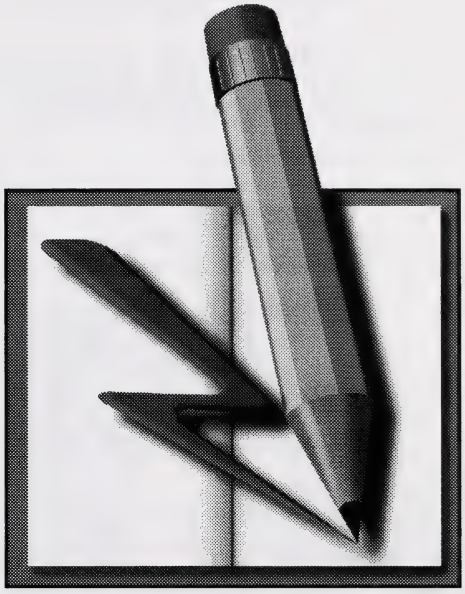
MATH FACTOR

EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Lesson 2: Exponential Growth and Decay

Student Workbook

Produced by

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This guide accompanies **Exponential Growth and Decay** from the Math Factor series.

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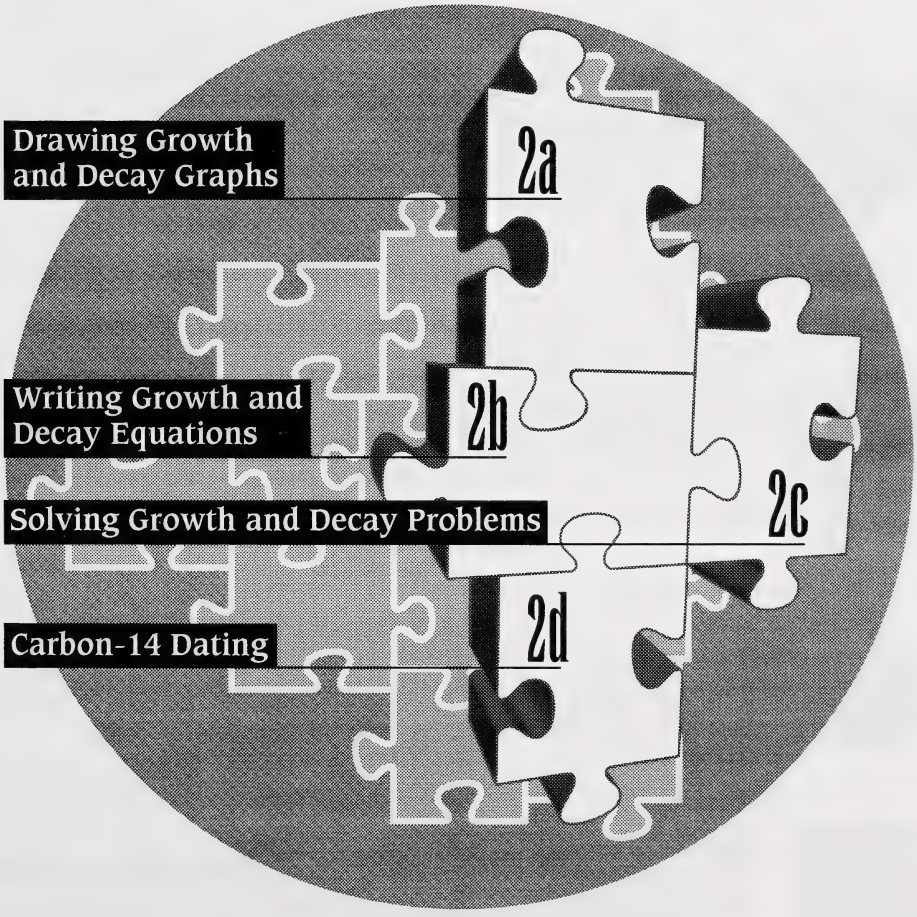
Lesson 2: EXPONENTIAL GROWTH AND DECAY

LEARNER EXPECTATIONS

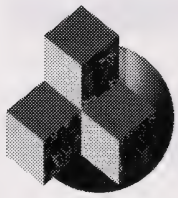
By the end of Lesson 2 you should be able to

- recognize exponential functions describing situations involving exponential growth and decay
- solve problems involving exponential growth and decay

This lesson has four parts:



EXPLANATION OF ICONS



Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Drawing Growth and Decay Graphs

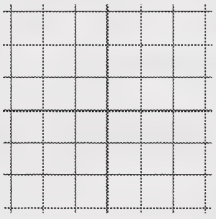
Watch the Introduction and Video Segment 2a.

Additional Examples
Mathematics 12: pp. 300-302
Mathematics: Principles and Process 12: p. 96
Holtmath 12: p.126 (Example 2)

Exercises

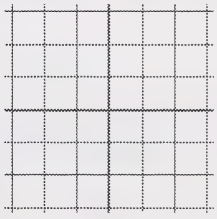
1. Sketch graphs that represent exponential growth and exponential decay on the following grids, and state all possible values of the base.

Exponential Growth



Possible value of the base:

Exponential Decay



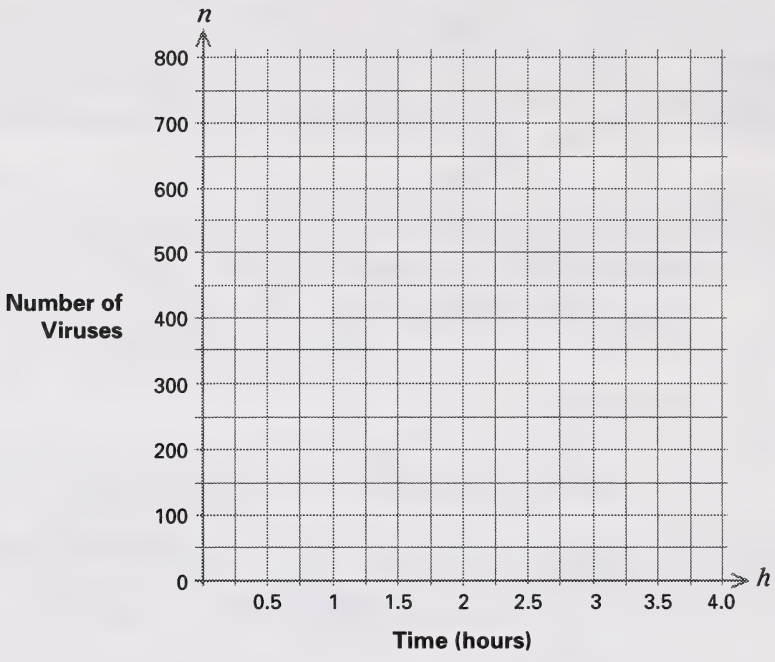
Possible value of the base:

2. A microbiologist finds that the population of a virus will decrease by half every thirty minutes when it is treated with a certain drug. There were eight hundred viruses in the initial sample.

   a. Complete the table of values for the first three hours after this virus was treated with the drug.

Time (hours)	0	0.5	1.0	1.5	2.0	2.5	3.0
Number of Viruses	800						

b. Construct a graph from the previous data. Draw a smooth continuous curve through the points.



c. Using the graph drawn in question 2b, estimate the number of viruses remaining after the following:

i. 15 min

ii. 2 h 45 min

d. Explain whether or not you think the number of viruses will eventually reach zero.

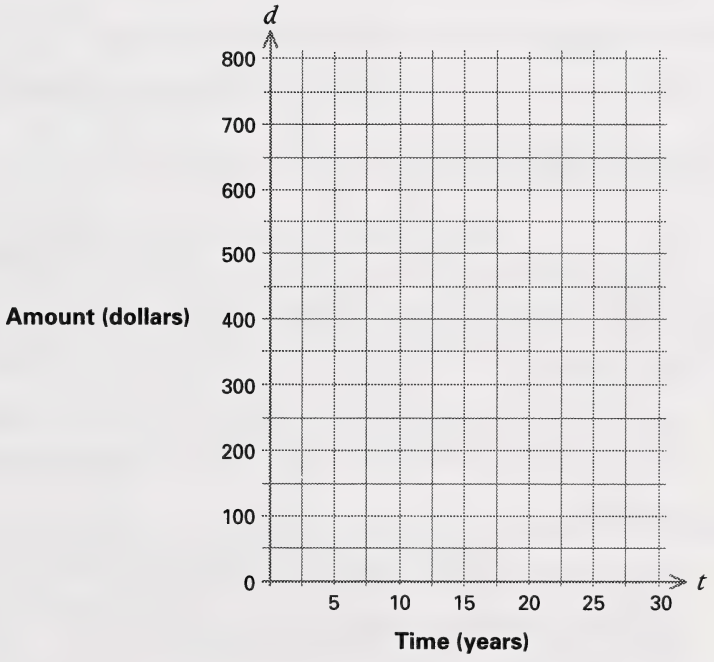
3. $100 was invested in a term deposit that compounded semi-annually and paid 8% per annum (per year). The amount the investment is worth at any time is represented by the formula $A = 100(1.04)^{2t}$, where t is the time (in years) the money is invested.

   Note: 8% is paid per year, but 4% would be paid every six months, and in one year there are two interest periods; thus, the exponent is $2t$.

   a. Complete the table of values, to the nearest dollar, for this investment.

Time (years)	0	5	10	15	20	25
Amount (dollars)	100					

   b. Draw the graph on the grid provided.



   c. From your graph, estimate the value of the original $100 investment after 12 years.

   d. When would the original investment triple in value?

EXERCISE 2a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 302 (Questions 1-8)
Mathematics: Principles and Process 12: p. 97 (Questions 1-4)

Writing Growth and Decay Equations

Challenge yourself

Fifty bacteria are present in an original sample and the bacteria double in population every 20 minutes. What equation or formula would describe this growth?

Watch Video Segment 2b.

All real-life situations that involve exponential growth or decay can be written in the following form:

$$f(x) = c(a)^x$$

The video mentions three examples: compound interest, doubling time and half-life.

Compound Interest

$A = P(1+i)^n$, where A is the final amount, P is the initial amount invested, i is the rate of interest per compounding period and n is the number of increases or compounding periods.

Example 1

Write an equation for the amount an investment will be worth, in terms of the time in years, if $1200 is invested at 10% per annum compounded quarterly.

Solution

Quarterly means interest is added or compounded four times a year. Thus, 10% per annum would pay 10% divided by 4, or 2.5% every compounding period; and the number of increases, n, would be four times the number of years, or $n = 4t$. Therefore, an equation for this investment, in terms of the time in years, would be

$A = 1200(1.025)^{4t}$.

Doubling Time

$P = P_0(2)^{\frac{t}{d}}$, where P is the final population, P_0 is the initial population, t is the time and d is the doubling time. Sometimes, population increase is expressed as a rate percent per year.

Example 2

The world population reached 5.4 billion in 1992 and was increasing at a rate of 1.6% each year. If this rate of growth is maintained, write an equation expressing the predicted population, P billion, as a function of the number of years since 1992.

Solution

$P = 5.4(1+0.016)^n$

5.4 billion is the initial population. 1 represents 100% and 0.016 is 1.6%. n is the number of years from 1992.
Simplified, $P = 5.4(1.016)^n$.

Half-Life

$M = M_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where M is the final mass, M_0 is the initial mass, t is the time and h is the half-life time.

Note: $M = M_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ is also equivalent to $M = M_0(2)^{-\frac{t}{h}}$ and is often written this way.

Example 3

For every metre a diver descends beneath the surface, the light intensity is reduced by 3%. Express the percent, P, of surface light present at any given depth (d metres) as a function of the depth.

Solution

$P = 100(1-0.03)^d$

100 represents 100% of the initial surface light. Subtract the 3% from 100% because the light is decreasing 3% for every metre below the surface.

Simplified, $P = 100(0.97)^d$.

Additional Examples
Mathematics 12: pp. 300-302

Exercises

4. Write equations to represent the following:

   a. There are 200 bees in a colony. The population, P, of the colony doubles every ten days. Express the population of the colony as a function of the time, t, in days.

   b. A town has a yearly growth rate of approximately 2.4%. The town's current population is 14 000. Write an equation to predict the town's population, P, in n years from now if the current rate of growth continues.

   c. As river water is passed through a filter, 10% of the unwanted matter is removed. Express the percent, P, of unwanted matter remaining as a function of the number, n, of filters used.

   d. $2000 is invested at 8.5% per annum compounded semi-annually. Write an equation that expresses the amount, A, the investment is worth in terms of the time, t, in years.

EXERCISE 2b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 303 (Questions 9-12)

Solving Growth and Decay Problems

A radioactive substance decays to $\frac{1}{4\sqrt{2}}$ of its original amount after 20 days. Find the half-life of this substance.

Watch Video Segment 2c.

Example

A laboratory sample initially contains 8000 g of radioactive substance. In four hours, there are 250 g of radioactive substance. Find the half-life period for this sample.

Solution

Find the half-life using $M = M_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ or $M = M_0(2)^{-\frac{t}{h}}$. From the question, the following can be determined:

- final mass: $M = 250$ g
- initial mass: $M_0 = 8000$ g
- time: $t = 4$ h

By substituting into the equation, we can solve for h, the half-life.

$$M = M_0(2)^{-\frac{t}{h}}$$

$$250 = 8000(2)^{-\frac{4}{h}}$$

$$\frac{250}{8000} = (2)^{-\frac{4}{h}} \qquad \text{(Divide both sides of the equation by 8000.)}$$

$$\frac{1}{32} = (2)^{-\frac{4}{h}} \qquad \text{(Reduce.)}$$

$$\frac{1}{2^5} = (2)^{-\frac{4}{h}} \qquad \text{(Express 32 as base 2.)}$$

$$2^{-5} = (2)^{-\frac{4}{h}} \qquad \left(\frac{1}{a^n} = a^{-n}\right)$$

$$\therefore -5 = \frac{-4}{h} \qquad \text{(If } x^m = x^n\text{, then } m = n.\text{)}$$

$$-5h = -4 \qquad \text{(Multiply both sides by } h.\text{)}$$

$$h = \frac{4}{5} \text{ or } 0.8 \text{ h}$$

This problem could have been done with time expressed in minutes. The answer would have been 48 min, which is equivalent to 0.8 h.

Additional Examples
Mathematics 12: p. 305 (Example 1)
Mathematics: Principles and Process 12: pp. 98, 101
Holtmath 12: p. 122 (Example 1)

Exercises

5. A technician notes that a certain bacteria has a doubling period of 4 h. If there is an initial amount of 500 bacteria, how many will there be in 24 h?

6. In nine days, an initial amount of radioactive substance decays to $\frac{1}{2\sqrt{2}}$ of its initial amount. Find the half-life of this radioactive substance.

7. A laboratory sample initially contains 120 bacteria. After 3 h, the sample grows to 1920 bacteria. Find the doubling period for this sample.

8. A radioactive substance has a half-life of 24 min.

   a. What percent of the original amount of the substance remains after 8 min?

   b. What percent of the original amount of the substance remains after 72 min?

9. A radioactive substance has a half-life of 4.6 days. Find the time required for 4 g of the substance to deteriorate to 1 g.

10. A certain kind of yeast will grow to 27 times its initial amount in 30 min. Find the time required for the original quantity to triple.

EXERCISE 2c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 308 (Questions 9-11)
Mathematics: Principles and Process 12: p. 98 (Questions 1-9), p. 103 (Questions 1-6)
Holtmath 12: p. 123 (Questions 3, 8)

Carbon-14 Dating

Watch Video Segment 2d.

As stated in the video, a formula used for carbon-14 dating is $A = A_0(0.5)^n$, where A is the amount of radioactivity in the given sample (e.g., rib of a mammoth), A_0 is the amount of radioactivity in a modern relative sample (e.g., rib of an elephant) and n is the number of half-lives. The half-life of carbon-14 is approximately 5730 years.

Example 1 provides more details to clarify Video Segment 2d.

Example 1

Recall that the sample, A (mammoth rib), has 25% as much radioactivity as a related modern sample, A_0 (modern elephant rib). Find the number of half-lives of the sample.

Solution

$$A = A_0(0.5)^n$$

$$25\% = 100\%(0.5)^n$$

$$\frac{25\%}{100\%} = (0.5)^n$$

$$\frac{1}{4} = (0.5)^n$$

$$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^n \qquad \text{(Express with the base } \tfrac{1}{2}\text{.)}$$

$$2 = n$$

Note: The half-life formula previously given could be used here: $M = M_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

Therefore, the sample has lived through two half-lives or two times 5730 years or 11 470 years.

Example 2

The approximate age of a piece of charcoal found at Stonehenge is thought to be 4000 years old. What percent of radioactivity should be expected from a sample of this charcoal compared to a modern piece of charcoal if, indeed, the charcoal is 4000 years old?

Solution

Since the sample is supposed to be 4000 years old, then the number of half-lives it would have gone through is $n = \frac{4000 \text{ years}}{5730 \text{ years}} = 0.7$ half-lives.

$$\therefore A = A_0 (0.5)^n$$
$$= 100\%(0.5)^{0.7}$$
$$= 100\%(0.62)$$
$$= 62\%$$

Use your calculator to evaluate $(0.5)^{0.7}$.

Therefore, the sample should have 62% as much radioactivity as a related modern sample if it is 4000 years old.

Additional Practice
Mathematics 12: p. 320
Mathematics: Principles and Process 12: p. 103

Exercises

11. For the following questions, assume that the half-life of carbon-14 is 5730 years.

    a. In a recent dig, a skeleton was found. Upon analysis, it was found that the amount of carbon-14 in a sample of the skeleton had decayed to $\frac{1}{8}$ of its original amount. How old was the skeleton?

    b. A piece of fossilized wood was analyzed. The percent of carbon-14 remaining is about 6%. Calculate the approximate age of the wood (to the nearest hundred years).

c. A fossilized leaf was found in a cave that was formed around or during the last ice age. Scientists thought the leaf was approximately 14 000 years old. What percent of radioactivity should the scientists expect the leaf to have compared to a modern relative leaf if it really is 14 000 years old?

EXERCISE 2d SOLUTIONS are at the back of this workbook.

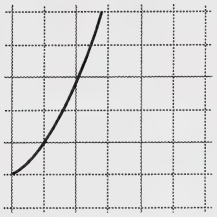
Additional Practice
Mathematics 12: p. 320 (Questions 1-3)
Mathematics: Principles and Process 12: p. 103 (Questions 7-9)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Exponential growth and decay are important applications of the exponential function. It is also necessary to understand how these formulas are derived and solved because they will be used in a later lesson. Before this can be done, you should master the concept in **Lesson 3: Inverse of the Exponential Function: The Logarithmic Function**.

Solutions

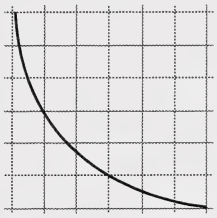
EXERCISE 2a

1. **Exponential Growth**



Possible value of the base: base > 1

Exponential Decay

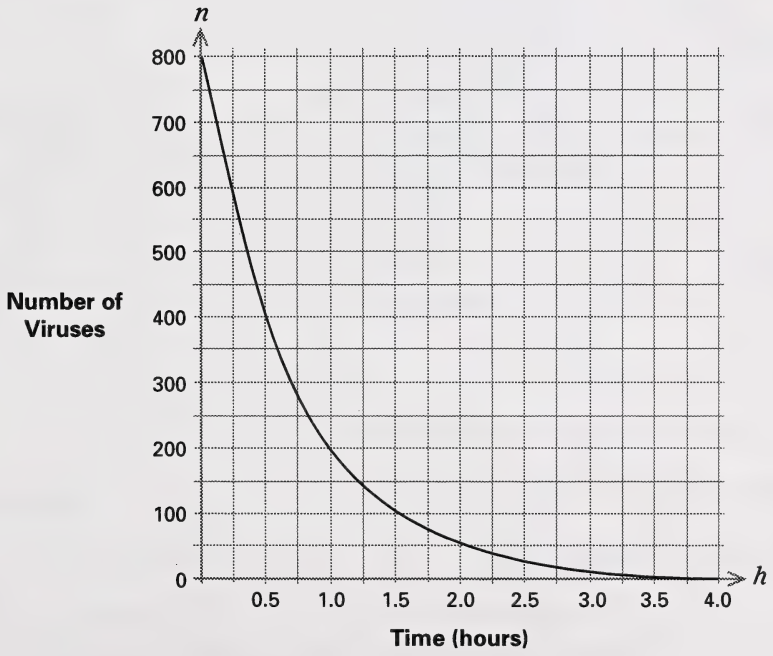


Possible value of the base: 0 < base < 1

2. a.

Time (hours)	0	0.5	1.0	1.5	2.0	2.5	3.0
Number of Viruses	800	**400**	**200**	**100**	**50**	**25**	**12.5**

b.



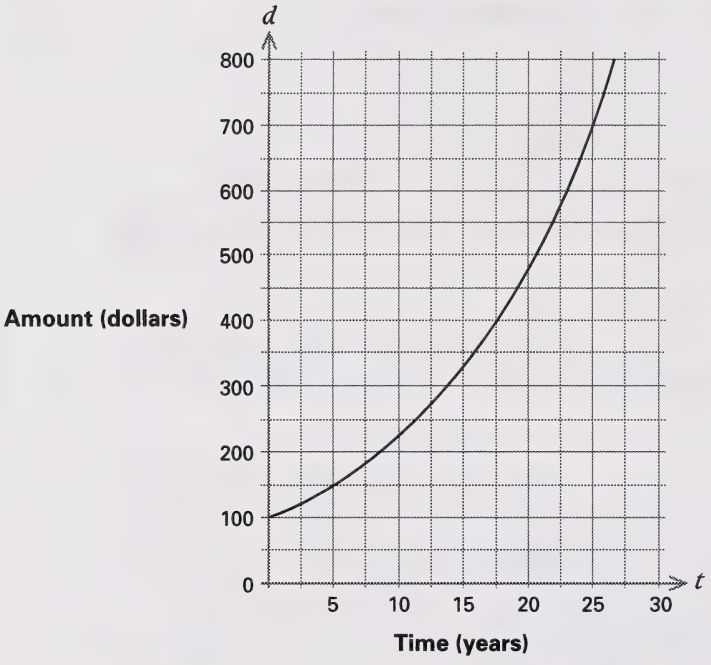
c. i. approximately 570 viruses ii. approximately 18 viruses

d. The number of viruses will probably eventually reach zero because a whole virus must die (one half cannot exist).

3. a.

Time (years)	0	5	10	15	20	25
Amount (dollars)	100	148	219	324	500	710

b.



c. approximately \$250

d. approximately 14 years

EXERCISE 2b

4. a. $P = 200(2)^{\frac{t}{10}}$, where t is time in days

b. $P = 14\,000(1.024)^{n}$, where n is number of years from now

c. $P = 100(0.9)^{n}$, where n is the number of filters used

d. $A = 2000(1.0425)^{2t}$, where t is time in years

EXERCISE 2c

5. $P = P_0(2)^{\frac{t}{d}}$

$= 500(2)^{\frac{24}{4}}$

$= 500(2)^{6}$

$= 32\,000$ bacteria

6. $M = M_0(2)^{-\frac{t}{h}}$

$\frac{1}{2\sqrt{2}} = (2)^{-\frac{9}{h}}$

$2^{-\frac{3}{2}} = 2^{-\frac{9}{h}}$

$\therefore -\frac{3}{2} = -\frac{9}{h}$

$-3h = -18$

$h = 6$ d

7. $P = P_0(2)^{\frac{t}{d}}$

$1920 = 120(2)^{\frac{3}{d}}$

$\frac{1920}{120} = (2)^{\frac{3}{d}}$

$16 = (2)^{\frac{3}{d}}$

$2^4 = (2)^{\frac{3}{d}}$

$\therefore 4 = \frac{3}{d}$

$d = \frac{3}{4}$ h

8. a. $M = M_0(2)^{-\frac{t}{h}}$

$= 100(2)^{-\frac{8}{24}}$

$= 100(2)^{-\frac{1}{3}}$

$= 79\%$

b. $M = M_0(2)^{-\frac{t}{h}}$

$= 100(2)^{-\frac{72}{24}}$

$= 100(2)^{-3}$

$= 12.5\%$

9. $M = M_0(2)^{-\frac{t}{h}}$

$1 = 4(2)^{-\frac{t}{4.6}}$

$\frac{1}{4} = (2)^{-\frac{t}{4.6}}$

$2^{-2} = (2)^{-\frac{t}{4.6}}$

$\therefore -2 = -\frac{t}{4.6}$

$t = 2 \times 4.6$

$= 9.2$ d

10. $P = P_0(3)^{\frac{t}{d}}$

$27 = 3^{\frac{30}{t}}$

$3^3 = 3^{\frac{30}{t}}$

$\therefore 3 = \frac{30}{t}$

$t = 10$ min

EXERCISE 2d

11. a. $A = A_0(0.5)^n$

$\frac{1}{8} = 1(0.5)^n$

$2^{-3} = 2^{-n}$

$\therefore -3 = -n$

$n = 3$ half-lives

or $3 \times 5730 = 17\,190$ years

b. $A = A_0(0.5)^n$

$0.06 = (0.5)^n$

$\log 0.06 = n \log 0.5$

$\frac{\log 0.06}{\log 0.5} = n$

$\therefore n = 4.059$

or $4.059 \times 5730 \doteq 23\,300$ years

c. $\frac{14\,000}{5730} = 2.4432$ half-lives

$A = A_0(0.5)^n$

$= 100(0.5)^{2.4432}$

$= 18\%$

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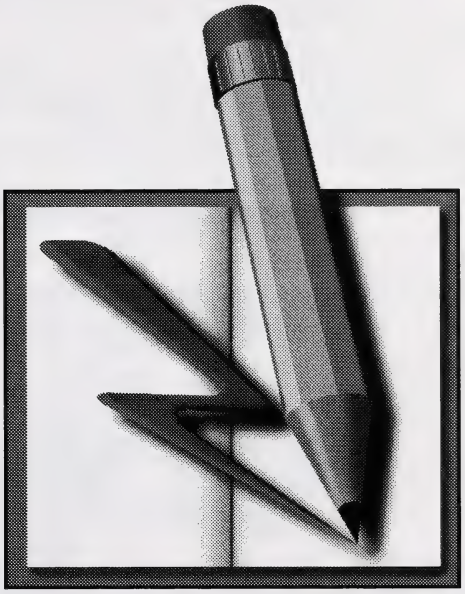
MATH FACTOR

EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Lesson 3:
Inverse of the Exponential Function: The Logarithmic Function

Student Workbook

Produced by

THE EDUCATION STATION ACCESS

for

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This guide accompanies **Inverse of the Exponential Function: The Logarithmic Function** from the Math Factor series.

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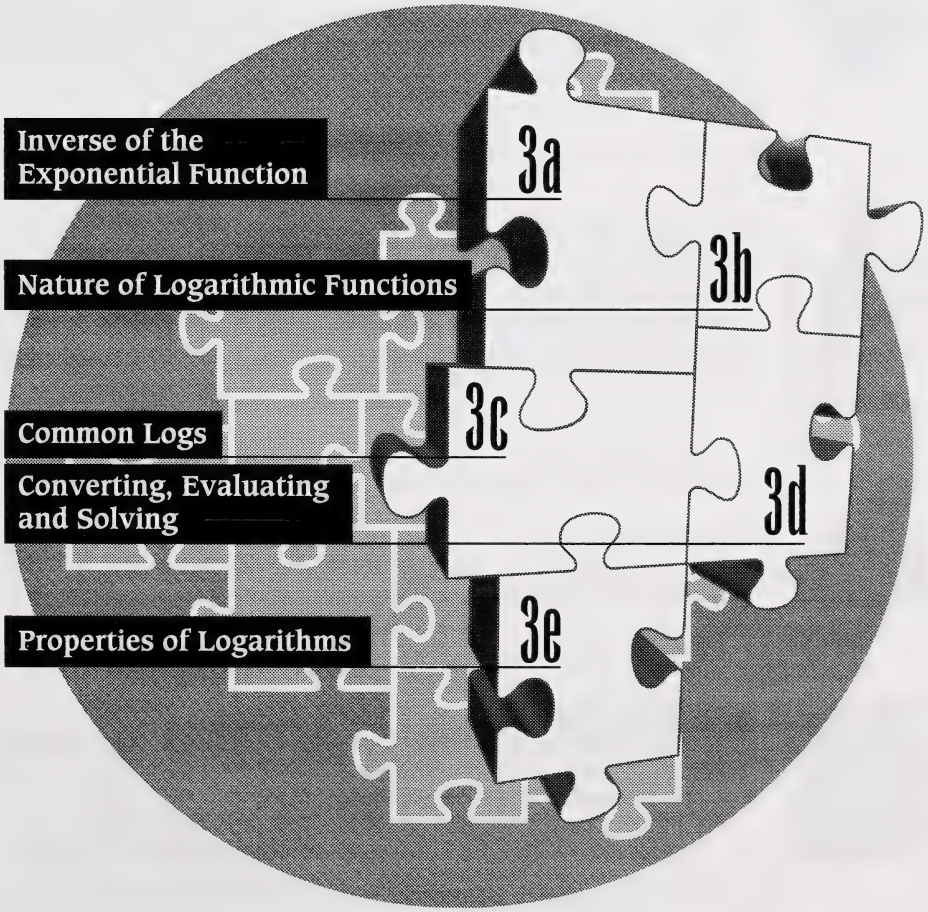
Lesson 3: INVERSE OF THE EXPONENTIAL FUNCTION: THE LOGARITHMIC FUNCTION

LEARNER EXPECTATIONS

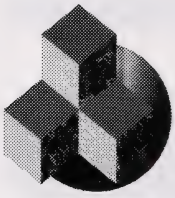
By the end of Lesson 3 you should be able to

- draw graphs of logarithmic functions as inverses of exponential functions
- use graphs of logarithmic functions to find values of one variable given the other
- draw graphs of logarithmic functions and analyze them using calculators or computers
- determine the domain and range of logarithmic functions
- convert functions between exponential form and logarithmic form

This lesson has five parts:



EXPLANATION OF ICONS



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Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

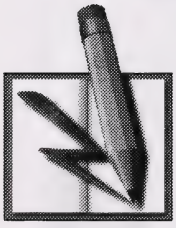
Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.



Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Inverse of the Exponential Function

Watch the Introduction and Video Segment 3a.

Example

Sketch the graph of $y = f(x)$, where $f(x) = 3^x$, and its inverse.

Solution

Step 1: Construct a table of values for the graph $y = 3^x$; then interchange the ordered pairs in the table of values for the inverse.

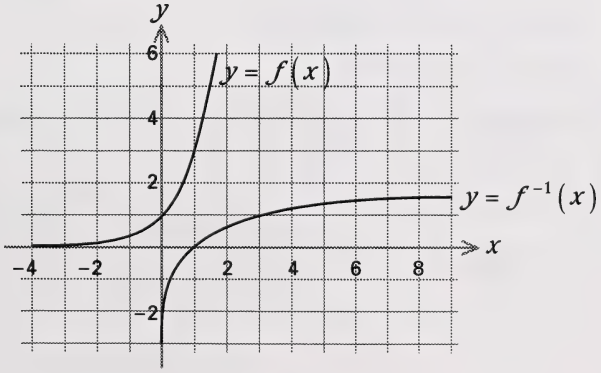
Function
$f(x)$ or $y = 3^x$

x	y
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

Inverse
$f^{-1}(x)$ or $x = 3^y$

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

Step 2: Plot the ordered pairs on a coordinate system and join the points as a continuous curve.



The equation $x = 3^y$ is equivalent to $y = \log_3 x$.

Note the following for the function and the inverse:

	Function $y = 3^x$ or $f(x) = 3^x$	**Inverse** $x = 3^y$ or $f^{-1}(x) = \log_3 x$
Domain	$x \in R$	$x > 0$
Range	$y > 0$	$y \in R$
***x*-intercept**	none	1
***y*-intercept**	1	none

Additional Examples
Mathematics 12: pp. 338, 339
Mathematics: Principles and Process 12: p. 110
Holtmath 12: p. 129 (Examples 1, 2)

Exercises

1. Write the inverse of each function in exponential form and logarithmic form.

   a. $f(x) = 5^x$ b. $f(x) = 10^x$ c. $g(x) = \left(\frac{1}{2}\right)^x$

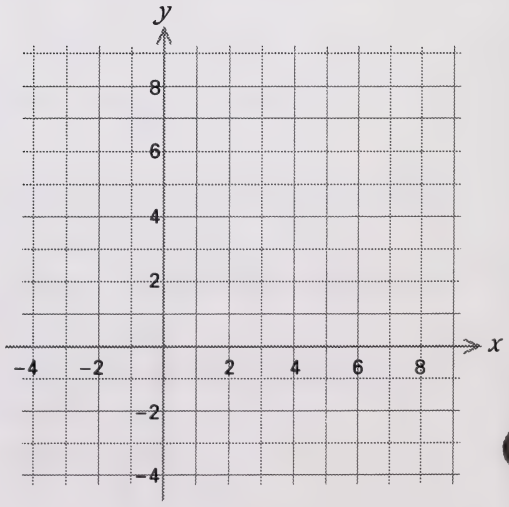
2. Given $y = f(x)$ where $f(x) = 4^x$:

   a. Complete the table of values and graph the function on the grid provided.

x	−2	−1	0	1	2
y					

   b. Complete the table of values for the inverse of $f(x)$ and graph the inverse on the same grid.

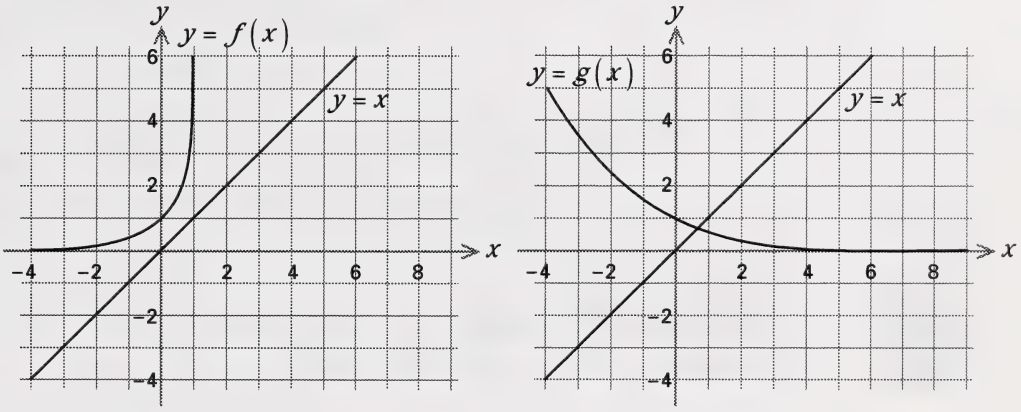
x					
y	−2	−1	0	1	2



c. Write the inverse in logarithmic and exponential form.

d. State the domain, range and intercepts for both graphs.

3. a. The graphs of $y = f(x)$ where $f(x) = 5^x$, $y = g(x)$ where $g(x) = \left(\frac{2}{3}\right)^x$, and $y = x$ are shown on the following grids. Sketch the inverse of each function on its grid by reflecting the functions about the line $y = x$.



b. Write the inverse of both functions in logarithmic and exponential form.

EXERCISE 3a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 340 (Questions 1-3)
Mathematics: Principles and Process 12: p. 110 (Questions 1-3)
Holtmath 12: p. 131 (Questions 3, 4)

Nature of Logarithmic Functions

Watch Video Segment 3b.

Example

The inverse of an exponential function can be written in two ways: exponential form and logarithmic form.

The following table shows how the function $f(x) = b^x$, where $b > 0$, and its inverse are related.

Exponential Function	Inverse of Exponential Function – Logarithmic Function	
	Logarithmic Form	Exponential Form
$f(x) = b^x$ or $y = b^x$	$f^{-1}(x) = \log_b x$ or $y = \log_b x$	$x = b^y$

The logarithm of a number is the exponent when the number is expressed as the power of an indicated base.

Additional Examples
Mathematics 12: pp. 338-340
Mathematics: Principles and Process 12: p. 110

Exercises

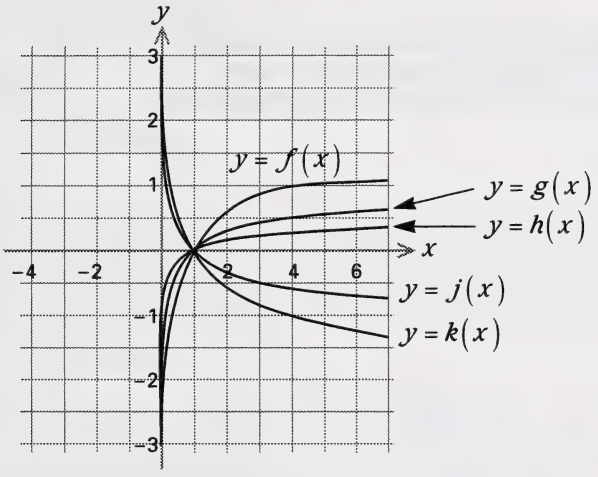
4. Write the following in exponential form.

   a. $\log_2 x = y$ b. $y = \log_{\frac{1}{3}} x$ c. $\log_b x = y$

5. Write the following in logarithmic form.

   a. $2^{-4} = \frac{1}{16}$ b. $25^{\frac{3}{2}} = 125$ c. $a^c = b$

6. Study the following examples of the graphs of the logarithmic functions $y = f(x)$, $y = g(x)$, $y = h(x)$, $y = j(x)$ and $y = k(x)$, where $f(x) = \log_2 x$, $g(x) = \log_5 x$, $h(x) = \log_{10} x$, $j(x) = \log_{\frac{1}{4}} x$ and $k(x) = \log_{\frac{1}{2}} x$; then answer the questions that follow.



a. What is the domain for all of these logarithmic functions?

b. What is the range for all of these logarithmic functions?

c. What point do all logarithmic functions appear to have in common?

d. If the base of the logarithm is greater than 1, what happens to the graph of the function?

e. If the base of the logarithm is between 0 and 1, what happens to the graph of the function?

7. Use the graphs in question 6 to estimate the following:

a. $\log_2 12$ b. $\log_5 10$ c. $\log_{10} 13$

d. $\log_{\frac{1}{2}} 7$ e. $\log_{\frac{1}{4}} 2.5$ f. $\log_2 0.5$

EXERCISE 3b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 341 (Question 6)
Mathematics: Principles and Process 12: p. 111 (Questions 4, 5)
Holtmath 12: p. 131 (Questions 9-11)

Common Logs

Challenge yourself

Determine the value of log 64.

MATH FACTOR

Watch Video Segment 3c.

The base of a logarithm can be any real number greater than zero. A common log is a logarithm with a base 10. Common logs are useful, as the number system is also base 10. The log button on your calculator means $\log_{10}$, which is read **log base 10**. This button can only be used when working with base 10 or a common log.

$$\log_{10} x = \log x$$

Note: If there is no base indicated, assume base 10.

The common log of any number can be found by estimating (as shown in the video) or by using a calculator. For example,

$$\log 4082 = \log_{10} 4082 \doteq 3.6109$$

$10^3 = 1000$

$10^4 = 10\,000$

$\therefore 10^{3.6109} \doteq 4082$

A common logarithm is the exponent when a number is expressed as a power of 10.

Given a common log, the number can also be found. For example,

$\log x \doteq 1.6990 \longrightarrow 10^{1.6990} = x$ (written in exponential form)

$x \doteq 50$

Calculators can also be used to find the number. Since the inverse of a logarithmic function is the exponential function, the inverse log key or the second function log key can be used.

TEXT

Additional Examples
Mathematics 12: pp. 321, 322
Mathematics: Principles and Process 12: p. 130 (Example 1)

Exercises

8. Use a calculator to determine the approximate value of each of the following. Record your answers to the nearest thousandth.

   a. log 24 b. log 0.24 c. log 2400

9. Determine the approximate value of x in each of the following equations.

   a. $\log x = 0.4771$

   b. $\log x = 1.3010$

   c. $\log x = -0.7782$

EXERCISE 3c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 323 (Questions 1, 2, 8, 10)
Mathematics: Principles and Process 12: p.131 (Questions 1, 2)
Holtmath 12: p. 134 (Question 3)

Converting, Evaluating and Solving

Challenge yourself

Evaluate the expression $\log_{27} 81$.

Watch Video Segment 3d.

Logarithms can be evaluated by expressing in exponential form.

Logarithmic Form $\Leftrightarrow$ Exponential Form

$\log_a b = c \Leftrightarrow a^c = b$

$\log_{\text{base}}$ number = exponent $\Leftrightarrow$ base$^{\text{exponent}}$ = number

Example 1

Evaluate $\log_6 \frac{1}{36}$.

Solution

$\log_6 \frac{1}{36} = x$ (Let the expression equal x.)

$6^x = \frac{1}{36}$ (Express in exponential form.)

$6^x = 6^{-2}$ (Express the equation with common bases.)

$x = -2$

$\therefore \log_6 \frac{1}{36} = -2$

Example 2

Solve the equation $\log_x \sqrt[3]{7} = \frac{1}{3}$.

Solution

$x^{\frac{1}{3}} = \sqrt[3]{7}$ (Express in exponential form.)

$x^{\frac{1}{3}} = 7^{\frac{1}{3}}$ (Express the equation with common exponents.)

$x = 7$

$\therefore \log_7 \sqrt[3]{7} = \frac{1}{3}$

Additional Examples
Mathematics 12: pp. 342, 343
Mathematics: Principles and Process 12: pp.112, 115
Holtmath 12: p. 130 (Example 3)

Exercises

10. Write the following in exponential form.

    a. $\log_3 81 = 4$

    b. $\log_{\frac{1}{2}} 64 = -6$

11. Write the following in logarithmic form.

    a. $2^{\frac{5}{2}} = 4\sqrt{2}$

    b. $\left(\frac{1}{3}\right)^{-4} = 81$

12. Evaluate the following:

    a. $\log_{32} 64$

    b. $\log_5 \frac{1}{125}$

13. Solve the following:

    a. $\log_x 4 = -\frac{1}{3}$

    b. $\log_x \sqrt{5} = \frac{1}{2}$

    c. $\log_2 x = -\frac{2}{3}$

    d. $\log x = 4.8$

14. Given $f(x) = a^x$, write the following:

    a. the inverse of the function in exponential form

    b. the inverse of the function in logarithmic form

15. "A logarithm is simply an exponent." What do you find when you determine the logarithm of a number?

EXERCISE 3d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 343, p. 344 (Questions 1, 2, 6-9)
Mathematics: Principles and Process 12: p. 115, p. 116 (Questions 1, 4)
Holtmath 12: p. 130 (Questions 1, 2)

Properties of Logarithms

Watch Video Segment 3e.

Summary

- $\log_a 1 = 0 \quad a > 0,\ a \neq 1$ The log of 1 for any base is 0, as $a^0 = 1$.
- $\log_a \left(a^x\right) = x$ Proof: If $y = a^x$,
  then $x = \log_a y$ (written in log form)
  or $x = \log_a \left(a^x\right)$ (Replace y with a^x.)
- $a^{\log_a x} = x$ Proof: If $y = \log_a x$,
  then $x = a^y$ (written in exponential form)
  or $x = a^{\log_a x}$ (Replace y with $\log_a x$.)
- An additional property is $\log_a a = 1$ because $a^1 = 1$.

Example 1

Evaluate $\log_3 1$.

Solution

$\log_3 1 = 0$

Example 2

Evaluate $\log_2 2$.

Solution

$\log_2 2 = 1$

Example 3

Evaluate $\log_5 5^{\frac{3}{4}}$.

Solution

$$\log_5 5^{\frac{3}{4}} = \frac{3}{4}$$

Example 4

Evaluate $3^{\log_3 7}$.

Solution

$$3^{\log_3 7} = 7$$

Example 5

Evaluate $\log_2 (\log_2 4)$.

Solution

$$\log_2 (\log_2 4) = \log_2 (\log_2 2^2) \qquad (\log_2 2^2 = 2)$$
$$= \log_2 2$$
$$= 1$$

Additional Examples
Mathematics 12: p. 343 (Example 3)
Mathematics: Principles and Process 12: p. 115

Exercises

16. Evaluate the following expressions.

a. $\log_{\frac{2}{3}} \frac{2}{3}$

b. $4^{\log_4 8}$

c. $\log_7 1$

d. $\log_5 5^6$

17. Find the value of x in each of the following equations.

a. $\log_4 x = 3$

b. $\log_9 x = \frac{1}{2}$

c. $\log_x 8 = 3$

d. $\log_5 5^{1.4} = x$

e. $\log_{36} x = -\frac{3}{2}$

f. $\log_x 49 = \frac{2}{3}$

g. $\log_4 64^{-1} = x$

h. $3^{\log_3\left(\frac{1}{9}\right)} = x$

18. If $\log_3 3 = 1$, show that $\log_3 3^n = n$.

19. Solve for x in each of the following equations.

a. $\log_6\left(\log_2 64\right) = x$

b. $4^{\log_4 64} = x$

c. $\log_x\left(\frac{1}{16}\right) = -\frac{2}{3}$

EXERCISE 3e SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 344 (Question 4)
Mathematics: Principles and Process 12: p. 116 (Questions 2, 3)
Holtmath 12: p. 131 (Question 12)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. You need to know how to draw the logarithmic function and how to convert from logarithmic form to exponential form and vice versa. You also need to understand the concept of a logarithm in order to do **Lesson 4: Laws of Logarithms: Evaluating Logarithmic Expressions**.

Solutions

EXERCISE 3a

1. a. Exponential Form: $x = 5^y$ b. Exponential Form: $x = 10^y$ c. Exponential Form: $x = \left(\frac{1}{2}\right)^y$

   Logarithmic Form: $y = \log_5 x$ Logarithmic Form: $y = \log_{10} x$ Logarithmic Form: $y = \log_{\frac{1}{2}} x$

2. a.

x	-2	-1	0	1	2
y	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

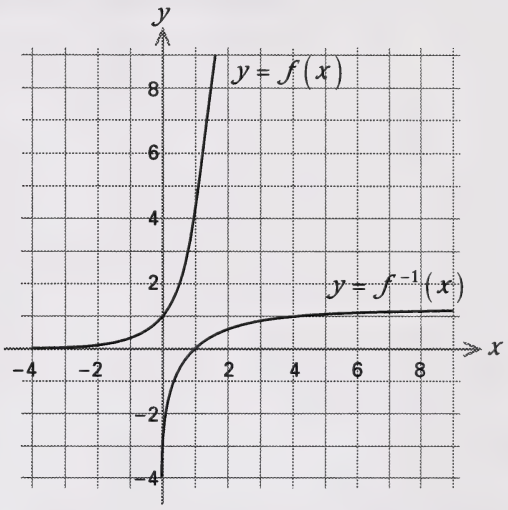
   b.

x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16
y	-2	-1	0	1	2

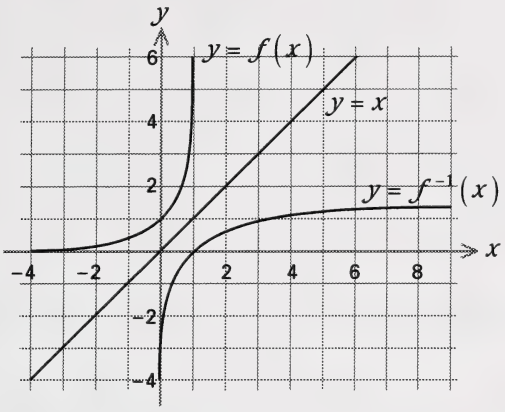
   c. Logarithmic Form: $y = \log_4 x$
   Exponential Form: $x = 4^y$

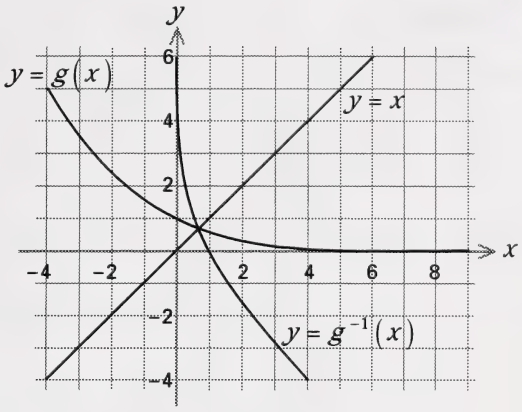
   d. Function: Domain: $\{x \mid x \in R\}$
   Range: $\{y \mid y > 0\}$
   x-intercept = none
   y-intercept = 1

   Inverse: Domain: $\{x \mid x > 0\}$
   Range: $\{y \mid y \in R\}$
   x-intercept = 1
   y-intercept = none



3. a.





b. $y = 5^x$

Logarithmic Form: $y = \log_5 x$

Exponential Form: $x = 5^y$

$y = \left(\frac{2}{3}\right)^x$

Logarithmic Form: $y = \log_{\frac{2}{3}} x$

Exponential Form: $x = \left(\frac{2}{3}\right)^y$

EXERCISE 3b

4. a. $2^y = x$ b. $\left(\frac{1}{3}\right)^y = x$ c. $b^y = x$

5. a. $\log_2\left(\frac{1}{16}\right) = -$ b. $\log_{25} 125 = \frac{3}{2}$ c. $\log_a b = c$

6. a. Domain: $\{x \mid x \geq 0\}$

b. Range: $\{y \mid y \in R\}$

c. They intersect at the point $(1, 0)$; there is no y-intercept; and the x-intercept = 1.

d. The graph is an increasing logarithmic curve. As x increases, so does y.

e. The graph is a decreasing logarithmic curve. As x increases, y decreases.

7. a. 3.4 b. 1.4 c. 1.1

d. −2.8 e. −0.7 f. −1

EXERCISE 3c

8. a. 1.380 b. −0.620 c. 3.380

9. a. 3 b. 20 c. 0.17

EXERCISE 3d

10. a. $3^4 = 81$ b. $\left(\frac{1}{2}\right)^{-6} = 64$

11. a. $\log_2 4\sqrt{2} = \frac{5}{2}$ b. $\log_{\frac{1}{3}} 81 = -4$

12. a. $\log_{32} 64 = x$
$32^x = 64$
$2^{5x} = 2^6$
$5x = 6$
$x = \frac{5}{6}$
$\therefore \log_{32} 64 = \frac{5}{6}$

b. $\log_5\left(\frac{1}{125}\right) = x$
$5^x = \frac{1}{125}$
$5^x = 5^{-3}$
$x = -3$
$\therefore \log_5\left(\frac{1}{125}\right) = -3$

13. a. $\log_x 4 = -\frac{1}{3}$
$x^{-\frac{1}{3}} = 4$
$\left(x^{-\frac{1}{3}}\right)^{-3} = 4^{-3}$
$x = \frac{1}{64}$

b. $\log_x \sqrt{5} = \frac{1}{2}$
$x^{\frac{1}{2}} = \sqrt{5}$
$\left(x^{\frac{1}{2}}\right)^2 = \left(\sqrt{5}\right)^2$
$x = 5$

c. $\log_2 x = -\frac{2}{3}$
$2^{-\frac{2}{3}} = x$
$x = \frac{1}{\sqrt[3]{4}}$

d. $\log x = 4.8$
$10^{4.8} = x$
$x \doteq 63\,096$

14. a. $x = a^y$ b. $f^{-1}(x) = \log_a x$

15. When you determine the log of a number, you find the exponent to which a specific base is raised in order to give the number.

EXERCISE 3e

16. a. $\log_{\frac{2}{3}} \frac{2}{3} = 1$ b. $4^{\log_4 8} = 8$ c. $\log_7 1 = 0$ d. $\log_5 5^6 = 6$

17. a. $\log_4 x = 3$
$4^3 = x$
$x = 64$

b. $\log_9 x = \frac{1}{2}$
$9^{\frac{1}{2}} = x$
$x = 3$

c. $\log_x 8 = 3$
$x^3 = 8$
$x = 2$

d. $\log_5 5^{1.4} = x$
$x = 1.4$

e. $\log_{36} x = -\frac{3}{2}$

$x = 36^{-\frac{3}{2}}$

$x = \frac{1}{216}$

f. $\log_x 49 = \frac{2}{3}$

$x^{\frac{2}{3}} = 49$

$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = (49)^{\frac{3}{2}}$

$x = 343$

g. $\log_4 64^{-1} = x$

$\log_4 \left(4^3\right)^{-1} = x$

$\log_4 4^{-3} = x$

$-3 = x$

h. $3^{\log_3\left(\frac{1}{9}\right)} = x$

$\log_3 x = \log_3 \frac{1}{9}$

$x = \frac{1}{9}$

18. $\log_3 3^n = n$ because $3^n = 3^n$ when the equation is written in exponential form.

19. a. $\log_6\left(\log_2 64\right) = x$

Since $\log_2 64 = 6$

$\log_6 6 = x$

$x = 1$

b. $4^{\log_4 64} = x$

$\log_4 x = \log_4 64$

$x = 64$

c. $\log_x \frac{1}{16} = -\frac{2}{3}$

$x^{-\frac{2}{3}} = \frac{1}{16}$

$\left(x^{-\frac{2}{3}}\right)^{-\frac{3}{2}} = \left(2^{-4}\right)^{-\frac{3}{2}}$

$x = 2^6$ or 64

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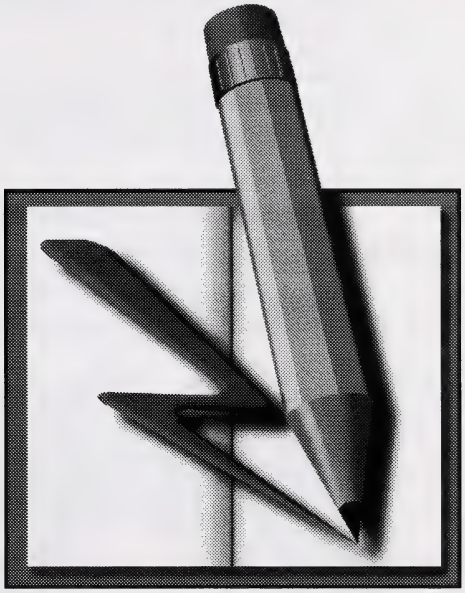

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Lesson 4:
Laws of Logarithms:
Evaluating Logarithmic Expressions

Student Workbook

Produced by

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This guide accompanies **Laws of Logarithms: Evaluating Logarithmic Expressions** from the Math Factor series.

Writer: Jerry Polei
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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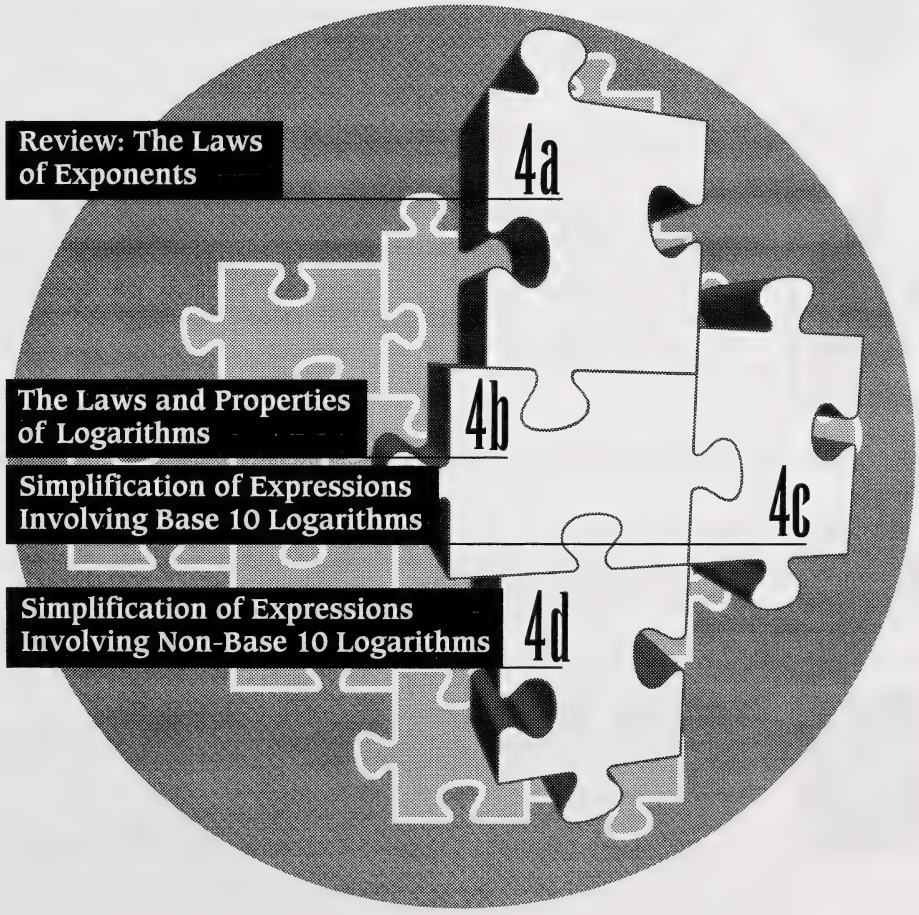
Lesson 4: LAWS OF LOGARITHMS: EVALUATING LOGARITHMIC EXPRESSIONS

LEARNER EXPECTATIONS

By the end of Lesson 4 you should be able to

- simplify and evaluate expressions using the laws of exponents
- state the laws and properties of logarithms
- recognize a common logarithm and apply the laws and properties of logarithms to simplify and evaluate common logarithmic expressions
- apply the laws and properties of logarithms to simplify and evaluate non-base 10 logarithmic expressions

This lesson has four parts:



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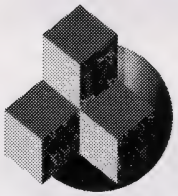

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Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

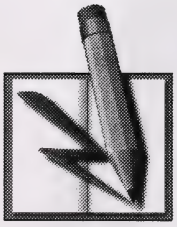
Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.



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Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Review: The Laws of Exponents

Watch the Introduction and Video Segment 4a.

Exponential Laws

Product Law: $a^m \times a^n = a^{m+n}$

Quotient Law: $\frac{a^m}{a^n} = a^{m-n}$

Power of a Power Law: $\left(a^m\right)^n = a^{mn}$

Power of a Product Law: $\left(ab\right)^m = a^m b^m$

Power of a Quotient Law: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

The **Product Law** and the **Quotient Law** can be used **only when the bases are the same**.

The following are some definitions related to exponents.

- $a^0 = 1$
- $a^{-n} = \frac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$

Note: For the case $a^{\frac{1}{n}} = \sqrt[n]{a}$, $a^{\frac{1}{n}}$ is the exponential form and $\sqrt[n]{a}$ is the radical form.

Example 1

Use the laws of exponents and the definitions to simplify the expression $3^4 \times \dfrac{3^3}{\left(3^3\right)^2}$. Express all powers with positive exponents.

Solution

$$3^4 \times \frac{3^3}{\left(3^3\right)^2} = \frac{3^{4+3}}{3^{3\times 2}}$$

$$= \frac{3^7}{3^6}$$

$$= 3^{7-6}$$

$$= 3^1$$

$$= 3$$

Example 2

Use the laws of exponents and the definitions to simplify the expression $\left(\dfrac{a^3b^2}{b^{-2}}\right)^4$. Express all powers with positive exponents.

Solution

$$\left(\frac{a^3b^2}{b^{-2}}\right)^4 = \frac{a^{12} \times b^8}{b^{-8}}$$

$$= a^{12} \times b^{8+8}$$

$$= a^{12}b^{16}$$

Example 3

Express $3^{\frac{1}{2}}$ in simplest radical form.

Solution

$$3^{\frac{1}{2}} = \sqrt{3}$$

Example 4

Express $9^{\frac{6}{5}}$ in simplest radical form.

Solution

$$9^{\frac{6}{5}} = \left(9^{\frac{1}{5}}\right)^6 \text{ or } \left(9^6\right)^{\frac{1}{5}}$$
$$= \left(\sqrt[5]{9}\right)^6 \text{ or } \sqrt[5]{\left(9^6\right)}$$

Example 5

Express $\frac{1}{\sqrt{x}}$ in simplest exponential form.

Solution

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}}$$
$$= x^{-\frac{1}{2}}$$

Example 6

Express $\left(\sqrt[3]{w}\right)\left(\sqrt[4]{w^3}\right)$ in simplest exponential form.

Solution

$$\left(\sqrt[3]{w}\right)\left(\sqrt[4]{w^3}\right) = \left(w^{\frac{1}{3}}\right)\left(w^{\frac{3}{4}}\right)$$
$$= \left(w^{\frac{4}{12}}\right)\left(w^{\frac{9}{12}}\right)$$
$$= w^{\frac{13}{12}}$$

Example 7

Simplify $\frac{5^3 \times (25)^4}{(625)^{\frac{1}{2}}}$ by first writing as powers of the same base.

Solution

$$\frac{5^3 \times (25)^4}{(625)^{\frac{1}{2}}} = \frac{5^3 \times \left(5^2\right)^4}{\left(5^4\right)^{\frac{1}{2}}} \qquad \left(25 = 5^2 \text{ and } 625 = 5^4\right)$$
$$= \frac{5^3 \times 5^8}{5^2}$$
$$= 5^{3+8-2}$$
$$= 5^9$$

Example 8

Simplify $9^{\frac{3}{2}} \times 27^{-\frac{2}{3}}$ by first writing as powers of the same base.

Solution

$$9^{\frac{3}{2}} \times 27^{-\frac{2}{3}} = \left(3^2\right)^{\frac{3}{2}} \times \left(3^3\right)^{-\frac{2}{3}}$$
$$= 3^{2\left(\frac{3}{2}\right)} \times 3^{3\left(-\frac{2}{3}\right)}$$
$$= 3^3 \times 3^{-2}$$
$$= 3$$

or

$$9^{\frac{3}{2}} \times 27^{-\frac{2}{3}} = \left(\sqrt{9}\right)^3 \times \left(\sqrt[3]{27}\right)^{-2}$$
$$= 3^3 \times 3^{-2}$$
$$= 3^1$$
$$= 3$$

Additional Examples
Mathematics 12: pp. 305, 306
Mathematics: Principles and Process 12: pp. 80, 82, 83, 86
Holtmath 12: pp. 116, 118, 120

Exercises

1. Use the laws of exponents to simplify the following expressions. Express your final answers with positive exponents.

a. $\dfrac{\left(3^2\right)^2}{\left(3^3 \times 3^2\right)}$

b. $\dfrac{\left(x^2yz\right)^2}{\left(xyz^2\right)^3} \times \left(\dfrac{2}{y}\right)^0$

c. $\left(\dfrac{x}{y}\right)^2 \left(\dfrac{y}{z}\right)^3 \left(\dfrac{y}{x}\right)^2$

d. $\left(t^{x-3}\right)\left(t^{2x+5}\right)$

2. Show that the expression on the left side is equivalent to the expression on the right side.

$$\frac{\left(3^{n+1}\right)\left(3^{n+3}\right)}{9^{n}} = 81$$

EXERCISE 4a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 307 (Questions 1, 3, 4), p. 308 (Question 14)
Mathematics: Principles and Process 12: p. 102 (Questions 1, 2)
Holtmath 12: p. 121 (Questions 1, 2, 4, 7)

The Laws and Properties of Logarithms

Watch Video Segment 4b.

Exercises

3. Complete the following statements by filling in the blanks.

   a. The Product Law of Logarithms states that $\log_a(xy) =$ _____.

   b. The Product Law of Logarithms indicates that the log of a product equals the sum of the log of each _____.

   c. The Quotient Law of Logarithms states that $\log_a\left(\frac{x}{y}\right) =$ _____.

   d. The Quotient Law of Logarithms indicates that the log of a quotient equals the _____ of the log of the numerator and the log of the denominator.

   e. The Power Law of Logarithms states that $\log_a x^m =$ _____.

   f. The Power Law of Logarithms indicates that the log of a power equals the _____ of the exponent and the log of the exponent's base.

   g. Applying the Power Law for Logarithms of Roots to $\log_a x^{\frac{1}{n}}$ allows you to say $\log_a x^{\frac{1}{n}} =$ _____.

   h. Since $\sqrt[n]{x} = x^{\frac{1}{n}}$, then $\log_a \sqrt[n]{x} = \log_a x^{\frac{1}{n}} =$ _____.

   i. Since $\left(\sqrt[n]{x}\right)^m = x^{\frac{m}{n}}$, then $\log_a \left(\sqrt[n]{x}\right)^m = \log_a x^{\frac{m}{n}} =$ _____.

   j. Since $\log_a \frac{1}{x} = \log_a x^{-1}$, $\log_a \frac{1}{x} =$ _____.

4. Each of the following statements applies a logarithmic law. In the space provided, state the law being applied.

   a. $\log_a 24 = \log_a 120 - \log_a 5$ _____

   b. $\log_a 24 = \log_a 8 + \log_a 3$ _____

   c. $\log_a 2^5 = 5\log_a 2$ _____

   d. $\log_a 8 = 3\log_a 2$ _____

   e. $\log_a \sqrt[5]{9} = \frac{1}{5}\log_a 9$ _____

   f. $\log_a \frac{1}{4} = -\log_a 4$ _____

EXERCISE 4b SOLUTIONS are at the back of this workbook.

Simplification of Expressions Involving Base 10 Logarithms

Rewrite the expression $\log a + 3\log b - \log c$ as a single logarithm.

Watch Video Segment 4c.

Example 1

Write the expression $\log\left[\dfrac{2^4 \times 3^7}{\left(\frac{1}{4}\right)}\right]$ as a sum and difference of logarithms.

Solution

$$\log\left[\frac{2^4 \times 3^7}{\left(\frac{1}{4}\right)}\right] = \log_{10}\left[\frac{2^4 \times 3^7}{\left(\frac{1}{4}\right)}\right]$$

$$= \log_{10}\left(2^4 \times 3^7\right) - \log_{10}\left(\frac{1}{4}\right)$$

Quotient Law of Logs

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$= \log_{10} 2^4 + \log_{10} 3^7 - \log_{10}\left(\frac{1}{4}\right)$$

Product Law of Logs

$$\log_a (xy) = \log_a x + \log_a y$$

$$= 4\log_{10} 2 + 7\log_{10} 3 - \left(-\log_{10} 4\right)$$

Power Law

$$\log_a\left(\frac{1}{x}\right) = \log_a (x)^{-1} = -\log_a x$$

$$= 4\log_{10} 2 + 7\log_{10} 3 + \log_{10} 4$$

Example 2

Use the logarithm laws to write the expression $2\log p + \frac{1}{3}\log 3 - 4\log p$ as a single logarithmic expression.

Solution

Write each term in the form $\log_{10} x$ or $\log x$.

$2\log p = \log p^2$

$\frac{1}{3}\log 3 = \log 3^{\frac{1}{3}}$

$4\log y = \log y^4$

$\therefore 2\log p + \frac{1}{3}\log 3 - 4\log y$

$= \log p^2 + \log 3^{\frac{1}{3}} - \log y^4$

$= \log\left(p^2 \times 3^{\frac{1}{3}}\right) - \log y^4$

$= \log \dfrac{p^2 \times 3^{\frac{1}{3}}}{y^4}$

Additional Examples
Mathematics 12: pp. 324-326
Mathematics: Principles and Process 12: p. 123 (Question 1a)

Exercises

5. Express the following as a single logarithmic expression and then evaluate.

   a. $\log 50 + \log 2$

   b. $\log 40 - \log 4$

   c. $2\log 5 + 2\log 2$

   d. $\log 8000 - 3\log 2$

   e. $\log 55.2 - \log 552$

6. Expand the following expressions using the logarithmic laws.

   a. $\log (mn)^3$

   b. $\log\left(\sqrt{k} \times \frac{\sqrt[3]{m}}{n^5}\right)$

7. Express the following as a single logarithmic expression.

   a. $\log x + \log y - \log 2$

   b. $3\log p - \log 2 - \frac{1}{3}\log s$

   c. $\log\left(\frac{1}{4}\right) + \log d$

EXERCISE 4c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 327 (Questions 1-7), p. 328 (Questions 8-14), p. 329 (Question 19)
Mathematics: Principles and Process 12: p. 122 (Question 6), p. 124 (Question 3d)
Holtmath 12: p. 134 (Questions 1d , 3, 4), p. 136 (Questions 2, 3)

Simplification of Expressions Involving Non-Base 10 Logarithms

Challenge yourself

If $\log_2 6 \doteq 2.584$, then solve $\log_2 48 = n$.

Watch Video Segment 4d.

Because calculators only evaluate common logs and natural logs, it is very important to be able to apply the logarithmic laws to rewrite logarithmic expressions using other bases.

Example 1

Evaluate the expression $\log_2 10 + \log_2 12.8$.

Solution

$$\begin{aligned}\log_2 10 + \log_2 12.8 &= \log_2 (10 \times 12.8) \\ &= \log_2 128 \\ &= \log_2 2^7 \\ &= 7\end{aligned}$$

Example 2

Given that $\log_3 2 \doteq 0.63$, find the value of $\log_3 (0.125)$.

Solution

$$\begin{aligned}\log_3 0.125 &= \log_3 \frac{125}{1000} \\ &= \log_3 \frac{1}{8} \\ &= -\log_3 8 \\ &= -\log_3 2^3 \\ &= -3\log_3 2 \\ &= -3 \times 0.63 \\ &= -1.89\end{aligned}$$

Additional Examples
Mathematics 12: pp. 346, 347
Mathematics: Principles and Process 12: pp. 119, 121, 123
Holtmath 12: p. 133, p. 134 (Question 4), p. 135

Exercises

8. Evaluate the following:

   a. $\log_3 \sqrt[3]{9}$

   b. $\log_4 (64)\left(\sqrt[5]{4}\right)$

   c. $\log_5 25\sqrt{125} + \log_3 81\left(\sqrt[3]{243}\right)$

9. If $\log_6 3 = 0.613$, then correct to the nearest thousandth, find the approximate value of each of the following expressions:

   a. $\log_6 \sqrt[4]{3}$

   b. $\log_6 \sqrt[4]{9}$

   c. $\log_6 108$

10. Write each of the following as a single logarithmic expression:

    a. $\ln A + 3\ln B - 4\ln C$

Note: ln indicates natural logarithms. That's base e. The same log laws apply to natural logs.

b. $\frac{1}{4}\log_b A + 3\log_b B - 3\log_b A$

c. $\frac{1}{7}\left(\log_5 A + 5\log_5 B\right) - 2\left(\log_5 C + 5\log_5 D\right)$

EXERCISE 4d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 349-350 (Questions 1-17)
Mathematics: Principles and Process 12: p. 120 (Questions 1-5), p. 122 (Questions 1-6), p. 124 (Questions 1-7)
Holtmath 12: p. 134 (Questions 1-8), p. 136 (Questions 1, 2, 4-9)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Before you continue on to **Lesson 5: Logarithmic and Exponential Equations**, it is important that you understand logarithmic laws and be able to apply them.

Solutions

EXERCISE 4a

1. a. $\frac{1}{3}$ b. $\frac{x}{yz^4}$ c. $\frac{y^3}{z^3}$ d. t^{3x+2}

2. $$\frac{\left(3^{n+1}\right)\left(3^{n+3}\right)}{9^n} = \frac{3^{2n+4}}{3^{2n}5^{3n}}$$
$$= 3^{2n+4-2n}$$
$$= 3^4$$
$$= 81$$

Therefore, the expression on the left side equals the expression on the right side.

EXERCISE 4b

3. a. $\log_a x + \log_a y$ b. factor c. $\log_a x - \log_a y$
   d. difference e. $m\log_a x$ f. product
   g. $\frac{1}{n}\log_a x$ h. $\frac{1}{n}\log_a x$ i. $\frac{m}{n}\log_a x$
   j. $-\log_a x$

4. a. Quotient Law b. Product Law c. Power Law
   d. Power Law e. Power Law f. Power Law

EXERCISE 4c

5. a. $\log\left(50\times 2\right) = \log 100$
$= 2$

   b. $\log\frac{40}{4} = \log 10$
$= 1$

   c. $\log\left(5^2\times 2^2\right) = \log 100$
$= 2$

   d. $\log\frac{8000}{8} = \log 1000$
$= 3$

   e. $\log\frac{55.2}{552} = \log\frac{1}{10}$
$= -1$

6. a. $\log m^3 + \log n^3 = 3\log m + 3\log n$

   b. $\log k^{\frac{1}{2}} + \log m^{\frac{1}{3}} - \log n^5 = \frac{1}{2}\log k + \frac{1}{3}\log m - 5\log n$

7. a. $\log\left(\frac{xy}{2}\right)$ b. $\log\left(\frac{p^3}{2s^{\frac{1}{3}}}\right)$ c. $\log\left(\frac{d}{4}\right)$

EXERCISE 4d

8. a. $\log_3 \sqrt[3]{9} = \log_3 \left(3^2\right)^{\frac{1}{3}}$
$= \log_3 3^{\frac{2}{3}}$
$= \frac{2}{3}$

b. $\log_4 (64)\left(\sqrt[5]{4}\right) = \log_4 \left(4^3\right)\left(4^{\frac{1}{5}}\right)$
$= \log_4 4^{3.2}$
$= 3.2$

c. $\log_5 25\sqrt{125} + \log_3 81\left(\sqrt[3]{243}\right) = \log_5 25 + \log_5 \sqrt{125} + \log_3 81 + \log_3 \sqrt[3]{243}$
$= \log_5 5^2 + \log_5 125^{\frac{1}{2}} + \log_3 3^4 + \log_3 243^{\frac{1}{3}} \quad \left(125 = 5^3 \text{ and } 243 = 3^5\right)$
$= 2\log_5 5 + \log_5 5^{\frac{3}{2}} + 4\log_3 3 + \log_3 3^{\frac{5}{3}}$
$= 2 + \frac{3}{2} + 4 + \frac{5}{3}$
$= 6 + \frac{19}{6}$
$= 9\frac{1}{6}$ or $\doteq 9.17$

9. a. $\log_6 \sqrt[4]{3} = \log_6 3^{\frac{1}{4}}$
$= \frac{1}{4}\log_6 3$
$\doteq 0.25 \times 0.613$
$\doteq 0.153$

b. $\log_6 \sqrt[4]{9} = \log_6 (9)^{\frac{1}{4}}$
$= \log_6 \left(3^2\right)^{\frac{1}{4}}$
$= \log_6 3^{\frac{1}{2}}$
$= \frac{1}{2}\log_6 3$
$\doteq 0.5 \times 0.613$
$\doteq 0.307$

c. $\log_6 108 = \log_6 (3 \times 36)$
$= \log_6 \left(3 \times 6^2\right)$
$= \log_6 3 + 2\log_6 6$
$\doteq 0.613 + 2$
$\doteq 2.613$

10. a. $\ln A + 3\ln B - 4\ln C = \ln A + \ln B^3 - \ln C^4$

$$= \ln\left(\frac{AB^3}{C}\right)$$

b. $\frac{1}{4}\log_b A + 3\log_b B - 3\log_b A = \log_b A^{\frac{1}{4}} + \log_b B^3 - \log_b A^3$

$$= \log_b\left(\frac{AB^3}{A^{\frac{11}{4}}}\right)$$

c. $\frac{1}{7}\left(\log_5 A + 5\log_5 B\right) - 2\left(\log_5 C + \log_5 D\right) = \frac{1}{7}\log_5\left(AB^5\right) - 2\log_5\left(CD\right)$

$$= \log_5\left(AB^5\right)^{\frac{1}{7}} - \log_5\left(CD\right)^2$$

$$= \log_5 A^{\frac{1}{7}} B^{\frac{5}{7}} - \log_5 C^2 D^2$$

$$= \log_5\left(\frac{A^{\frac{1}{7}} B^{\frac{5}{7}}}{C^2 D^2}\right)$$

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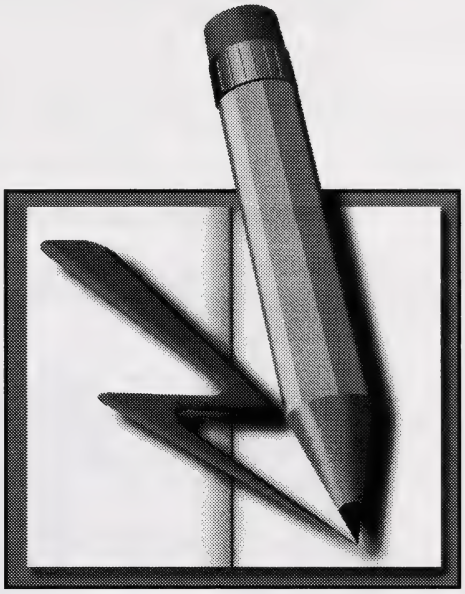
MATH FACTOR

EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Lesson 5: Logarithmic and Exponential Equations

Student Workbook

Produced by

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This guide accompanies **Logarithmic and Exponential Equations** from the Math Factor series.

Writer: Jerry Polei
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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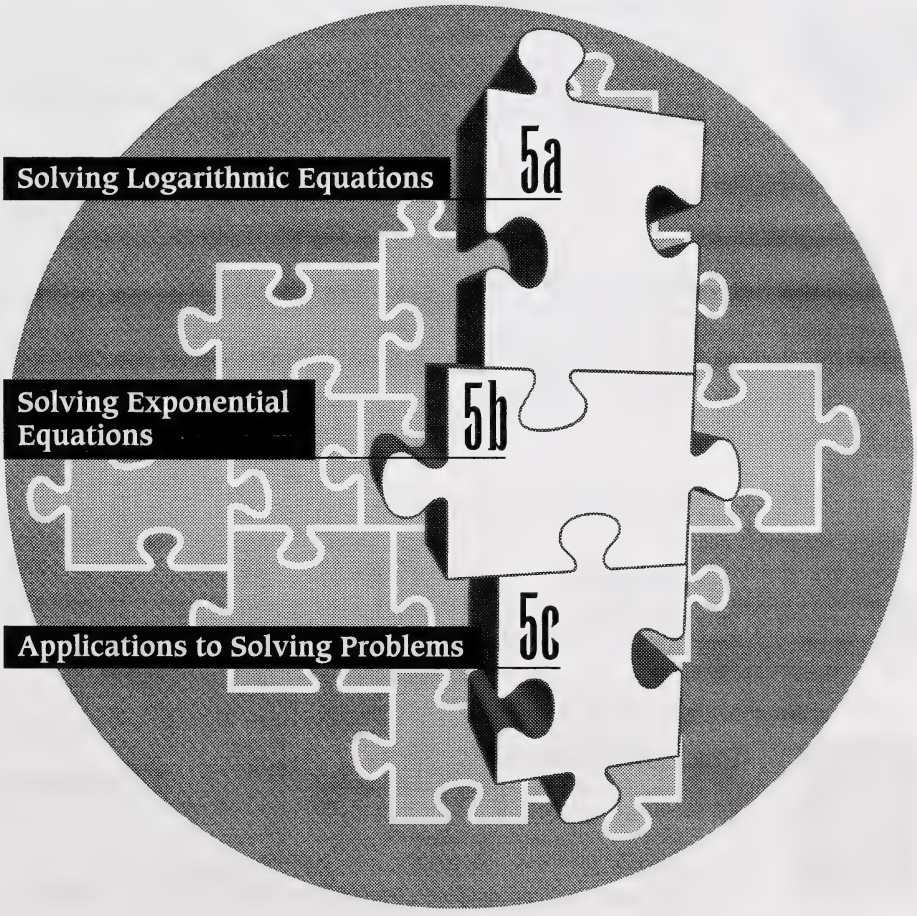
Lesson 5: LOGARITHMIC AND EXPONENTIAL EQUATIONS

LEARNER EXPECTATIONS

By the end of Lesson 5 you should be able to

- solve and verify logarithmic equations of any base
- solve and verify exponential equations using logarithms
- apply the logarithmic and exponential equation-solving skills to find solutions to problems involving population growth, radioactive decay and compound interest

This lesson has three parts:



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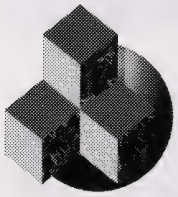


Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

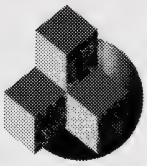
Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*



Building Blocks

Do you remember how to solve quadratic equations? If not, study the examples and do the exercises that follow before watching this video program.

To solve an equation or problem that involves logarithms, it may be necessary to solve a quadratic equation. A quadratic equation is a second-degree equation in one variable. It can be written in the form $ax^2 + bx + c = 0$ where a, b and c are real numbers and **a cannot equal 0.**

Example 1

Solve the quadratic equation $x^2 + 7x = 0$ by factoring.

Solution

$x^2 + 7x = 0$

$x(x+7) = 0$

$x = 0$ or $x + 7 = 0$

$x = 0$ or $x = -7$

Example 2

Solve the quadratic equation $2x^2 - x - 3 = 0$ by factoring.

Solution

$2x^2 - x - 3 = 0$

$(2x-3)(x+1) = 0$

$2x - 3 = 0$ or $x + 1 = 0$

$x = \frac{3}{2}$ or $x = -1$

Example 3

Solve the quadratic equation $x^2 + 2x = 1$ using the quadratic formula.

Solution

$x^2 + 2x = 1$

$x^2 + 2x - 1 = 0$

Because the equation cannot be factored, use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$a = 1,\ b = 2,\ c = -1$

Substitute the values for a, b and c as shown to obtain the following:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2 \times 1}$$
$$= \frac{-2 \pm \sqrt{8}}{2}$$
$$= \frac{-2 \pm 2\sqrt{2}}{2} \qquad \left(\sqrt{8} = 2\sqrt{2}\right)$$
$$= -1 \pm \sqrt{2}$$

The solutions are $x = -1 + \sqrt{2}$ or $x = -1 - \sqrt{2}$, which approximate to 0.414 and -2.414.

Additional Examples
Mathematics 12: p. 27 (Example 2), p. 28 (Example 6)
Mathematics: Principles and Process 12: pp. 48, 53
Holtmath 12: p. 62 (Example 3)

Exercises

1. Solve the following quadratic equations.

   a. $x^2 - 25 = 0$

   b. $4x^2 - 12x = 0$

   c. $9x^2 = 2x$

   d. $x^2 = 2x + 24$

e. $3x^2 - 10x + 8 = 0$

f. $2x^2 + 4x - 3 = 0$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Additional Exercises
Mathematics 12: p. 29 (Question 2), p. 30 (Question 6), p. 31 (Question 18)
Mathematics: Principles and Process 12: p. 50 (Question 7)
Holtmath 12: p. 63 (Questions 4b, 4c, 4g, 4j), p. 66 (Questions 2a, 2e)

Solving Logarithmic Equations

Challenge yourself

Solve $2\log x = \log 16$.

Watch the Introduction and Video Segment 5a.

Techniques for Solving Logarithmic Equations

- Check for restrictions on the solution. Remember, the log of a negative number or zero is undefined.
- Write coefficients of log terms as exponents; for example, $3\log x = \log x^3$.
- Write sums and differences of logs as the logs of products and quotients.

Example 1

Solve $\log x - \log 5 = \log 6x + \log 4x$.

Solution

The restriction on the value of x is $x > 0$.

Method 1

$$\log x - \log 6x - \log 4x = \log 5$$
$$\log \frac{x}{6x} - \log 4x = \log 5$$
$$\log \frac{1}{6} - \log 4x = \log 5$$
$$\log \frac{\frac{1}{6}}{4x} = \log 5$$
$$\log \frac{1}{24x} = \log 5$$
$$\log \frac{1}{24x} = \log 5$$
$$\frac{1}{24x} = 5$$
$$120x = 1$$
$$x = \frac{1}{120}$$

Method 2

$$\log x - \log 5 = \log 6x + \log 4x$$
$$\log \frac{x}{5} = \log\left((6x)(4x)\right)$$
$$\frac{x}{5} = 24x^2$$
$$x - 120x^2 = 0$$
$$x(1 - 120x) = 0$$
$$x = 0 \text{ or } x = \frac{1}{120}$$

Since $x > 0$, the solution to the equation is $x = \frac{1}{120}$.

Verification:

L.S.	R.S.
$\log x - \log 5$	$\log 6x + \log 4x$
$= \log \frac{1}{120} - \log 5$	$= \log(6)\left(\frac{1}{120}\right) + \log(4)\left(\frac{1}{120}\right)$
$= \log \frac{1}{120} - \log 5$	$= \log \frac{1}{20} + \log \frac{1}{30}$
$\doteq -2.08 - 0.70$	$\doteq -1.30 + -1.48$
$\doteq -2.78$	$\doteq -2.78$

L.S. = R.S.

Example 2

Solve $\log_4(2x-1) + \log_4(x+3) = 1$.

Solution

Both $2x-1$ and $x+3$ must be greater than 0. If $2x-1>0$, then $x > \frac{1}{2}$.
If $x+3>0$, then $x>-3$.

Since both expresssions must be positive, it follows that x must be greater than $\frac{1}{2}$.

$$\log_4(2x-1) + \log_4(x+3) = 1$$
$$\log_4(2x-1)(x+3) = 1$$
$$\log_4(2x-1)(x+3) = \log_4 4 \quad \text{or} \quad (2x-1)(x+3) = 4^1$$
$$(2x-1)(x+3) = 4$$
$$2x^2 + 5x - 3 = 4$$
$$2x^2 + 5x - 7 = 0$$
$$(2x+7)(x-1) = 0$$
$$2x+7 = 0 \quad \text{or} \quad x-1 = 0$$
$$x = -\frac{7}{2} \quad \text{or} \quad x = 1$$

Since x must be greater than $\frac{1}{2}$, the solution is 1.

Verification:
Substitute $x = 1$.

L.S.	R.S.
$\log_4(2\times 1 - 1) + \log_4(1+3)$	1
$= \log_4(1) + \log_4(4)$	
$= 0 + 1$	
$= 1$	$(4^0 = 1 \text{ and } 4^1 = 4)$

L.S. = R.S.

Example 3

Solve $\frac{1}{2}\log x - \frac{1}{3}\log 8x - \log 2 + 1 = \log 5$.

Solution

The restriction on the value of x is $x > 0$.

$$\frac{1}{2}\log x - \frac{1}{3}\log 8x = \log 5 + \log 2 - 1$$

$$\log x^{\frac{1}{2}} - \log(8x)^{\frac{1}{3}} = \log(2 \times 5) - 1$$

$$\log\left(\frac{x^{\frac{1}{2}}}{8^{\frac{1}{3}}\, x^{\frac{1}{3}}}\right) = \log(10) - 1$$

$$\log\frac{x^{\frac{1}{6}}}{2} = 1 - 1 \qquad \left(8^{\frac{1}{3}} = \sqrt[3]{8} = 2\right)$$

$$\log\frac{x^{\frac{1}{6}}}{2} = 0$$

$$\frac{x^{\frac{1}{6}}}{2} = 10^0$$

$$\frac{x^{\frac{1}{6}}}{2} = 1$$

$$x^{\frac{1}{6}} = 2$$

$$\left(x^{\frac{1}{6}}\right)^6 = 2^6$$

$$x = 64$$

Verification:

L.S.	R.S.
$\frac{1}{2}\log 64 - \frac{1}{3}\log 512 - \log 2 + 1$	$\log 5$
$= \log 64^{\frac{1}{2}} - \log 512^{\frac{1}{3}} - \log 2 + 1$	$\doteq 0.698\,97$
$= \log 8 - \log 8 - \log 2 + 1$	
$= 1 - \log 2$	
$\doteq 0.698\,97$	

L.S. = R.S.

Additional Examples
Mathematics 12: p. 348
Mathematics: Principles and Process 12: pp. 127, 129
Holtmath 12: p.140 (Example 2)

Exercises

1. Solve the following equations.

   a. $\log_3 x + \log_3 6 = \log_3 5$

   b. $4\log x = \log\frac{1}{16}$

c. $\log 6 - \log x = \log 3x + \log 2x$

d. $\log_5 2x - \log_5 7 = 1$

e. $\log_2 2x + \log_2 (x-1) = 2$

f. $2x - 1 = \log_4 8 + \log_4 4 - \log_4 0.5$

g. $\log \frac{2x}{5} - \log 8 = 2$

h. $\frac{1}{4}\log x - \frac{1}{2}\log 4x + 2 = 2\log 5 + 2\log 2$

EXERCISE 5a SOLUTIONS are at the back of this workbook.

Additional Exercises
Mathematics 12: p. 329 (Question 20), p. 350 (Questions 18, 19)
Mathematics: Principles and Process 12: pp. 127-128 (Questions 1-7), p.129 (Questions 1-4)
Holtmath 12: p. 141 (Questions 3, 6, 7, 9, 10)

Solving Exponential Equations

Solve $3^{2x} = 4^{5x-2}$.

Watch Video Segment 5b.

Example 1

Solve $4^{-2x} = \left(\frac{1}{8}\right)^{x+3}$.

Solution

$$\left(2^2\right)^{-2x} = \left(2^{-3}\right)^{x+3}$$
$$\left(2\right)^{-4x} = \left(2\right)^{-3x-9}$$
$$-4x = -3x - 9$$
$$-x = -9$$
$$x = 9$$

If the bases can be made the same, express both sides as powers of the same base. Equate the exponents; then solve.

Example 2

Solve $3^{x+2} = 2^{2x+2}$.

Solution

$$\log 3^{x+2} = \log 2^{2x+2}$$
$$(x+2)\log 3 = (2x+2)\log 2$$
$$x\log 3 + 2\log 3 = 2x\log 2 + 2\log 2$$
$$x\log 3 - 2x\log 2 = 2\log 2 - 2\log 3$$
$$x(\log 3 - 2\log 2) = 2\log 2 - 2\log 3$$
$$x = \frac{(2\log 2 - 2\log 3)}{(\log 3 - 2\log 2)}$$
$$x \doteq 2.82$$

If the bases cannot be made equal, take the common log of both sides; then use the Power Law of logs.

Example 3

A radioactive substance has a half-life of six days. What is the time required for 300 g of this substance to decay to 15 g?

Solution

The formula $M = M_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ represents the amount, M, remaining after time, t. The original amount is M_0, and h is the half-life of the substance.
Substitute $M_0 = 300$, $M = 15$ and $h = 6$ into this formula.

$$15 = 300\left(\frac{1}{2}\right)^{\frac{t}{6}}$$

$$\frac{1}{20} = \left(\frac{1}{2}\right)^{\frac{t}{6}}$$

$$\log\frac{1}{20} = \log\left(\frac{1}{2}\right)^{\frac{t}{6}}$$

$$\log\frac{1}{20} = \frac{t}{6}\log\frac{1}{2}$$

$$t = \frac{6\log\frac{1}{20}}{\log\frac{1}{2}}$$

$$\doteq 25.93$$

The answer can be quickly estimated.

After 1 half-life, 150 g remain.
After 2 half-lives, 75 g remain.
After 3 half-lives, 37.5 g remain.
After 4 half-lives, 18.75 g remain.

Just over 4 half-lives would be required (approximately 25 d).

Verification:

L.S.	R.S.
$300\left(\frac{1}{2}\right)^{\frac{25.93}{6}}$	15
$\doteq 15.0027$	

L.S. = R.S.

It would take approximately 26 d for the original 300 g to decay to 15 g.

Example 4

Solve $\log_3 7 = \log_2 x$, correct to three decimal places.

Solution

There are at least two ways that this can be solved. Method 1 uses the logarithmic laws that you have studied. Method 2 introduces a new formula called the **Change of Base Formula**.

Method 1

$$3^{\log_2 x} = 7$$
$$\log 3^{\log_2 x} = \log 7$$
$$\log_2 x(\log 3) = \log 7$$
$$\log_2 x = \frac{\log 7}{\log 3}$$
$$x = 2^{\frac{\log 7}{\log 3}}$$
$$x \doteq 3.413$$

Change of Base Formula

$\log_b a = \dfrac{\log_x a}{\log_x b}$, where x is any valid base

Method 2

Determine $\log_3 7$ correct to the nearest thousandth.

$$\log_3 7 = \frac{\log_{10} 7}{\log_{10} 3}$$
$$= \frac{\log 7}{\log 3}$$
$$\doteq 1.7712$$

Use this value in the original equation.

$$1.7712 \doteq \log_2 x$$
$$2^{1.7712} \doteq x$$
$$x \doteq 3.413$$

Additional Examples
Mathematics 12: p. 327 (Example 8)
Mathematics: Principles and Process 12: p. 133 (Example 2)
Holtmath 12: p. 139 (Examples 2, 3)

Exercises

2. Solve the following exponential equations, correct to the nearest thousandth.

a. $4^x = 9$

b. $4^{\frac{x}{4}} = 23$

c. $3^{2x} = 4^{5x-2}$

3. Evaluate the following expressions correct to the nearest thousandth.

a. $\log_4 9$

b. $\log_{\frac{1}{5}} 7$

c. $\dfrac{1}{\log_3 8}$

EXERCISE 5b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 328 (Questions 15-17), p. 344 (Question 13)
Mathematics: Principles and Process 12: p. 134 (Questions 6-9)
Holtmath 12: p.139 (Questions 4, 7, 8), p. 145 (Questions 4, 7, 8)

Applications to Solving Problems

Determine the time it would take for an investment to double given that the interest rate is 12% compounded annually.

Example

Find the approximate time required for \$7000 invested at 10% per annum to grow to \$10 000.

Solution

$$A = P(1+i)^n$$

$$10\,000 = 7000(1+0.1)^n$$

$$\frac{10}{7} = 1.1^n$$

$$\log\frac{10}{7} = \log 1.1^n$$

$$\log\frac{10}{7} = n\log 1.1$$

$$\frac{\log\frac{10}{7}}{\log 1.1} = n$$

$$n \doteq \frac{0.154\,90}{0.041\,39}$$

$$n \doteq 3.74$$

The time required is approximately 3.74 years.

Verification:

Substitute $t \doteq 3.74$ into $10\,000 = 7000(1.1)^n$.

$$10\,000 \doteq 7000(1.1)^{3.74}$$

$$10\,000 \doteq 9997.85$$

Additional Examples
Mathematics 12: p. 351 (Example 3), p. 352 (Example 5), p. 353 (Example 6)
Mathematics: Principles and Process 12: p. 135
Holtmath 12: p. 138

Exercises

4. Solve the following problems.

   a. Determine the number of years required for \$700 to triple if the \$700 is invested at 10% per annum compounded semi-annually.

   b. A radioactive substance has a half-life of 28 years. Calculate the time required for 40 g to decay to 4 g.

   c. The population of a country was 7.86×10^7 in 1970 and 8.03×10^7 in 1980. Predict what the population might be in the year 2000.

EXERCISE 5c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: pp. 354-355 (Questions 4, 5, 8-17), p. 360 (Questions 1-3)
Mathematics: Principles and Process 12: p. 136 (Questions 2, 6)
Holtmath 12: p. 141 (Question 8)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. The ability to solve logarithmic and exponential equations is an important skill. Proficiency in solving these kinds of equations will help you as you continue on to **Lesson 6: Describing Real – World Phenomena Using Logarithms**.

Solutions

BUILDING BLOCKS

1. a. $(x+5)(x-5)=0$ or $x^2=25$

$x=-5$ or $x=5$ $\qquad x=\pm\sqrt{25}$

$=\pm 5$

b. $4x(x-3x)=0$

$4x=0$ or $x-3=0$

$x=0$ or $x=3$

c. $9x^2-2x=0$

$x(9x-2)=0$

$x=0$ or $x=\frac{2}{9}$

d. $x^2-2x-24=0$

$(x-6)(x+4)=0$

$x=6$ or $x=-4$

e. $(3x-4)(x-2)=0$

$x=\frac{4}{3}$ or $x=2$

f. $x=\frac{-4\pm\sqrt{16-(4\times 2\times -3)}}{4}$

$=\frac{-4\pm\sqrt{40}}{4}$

$=\frac{-4\pm 2\sqrt{10}}{4}$ $\qquad \left(\sqrt{40}=2\sqrt{10}\right)$

$=\frac{-2\pm\sqrt{10}}{2}$

$x=\frac{-2+\sqrt{10}}{2}$ or $x=\frac{-2-\sqrt{10}}{2}$

$x\doteq 0.581\,14$ or $x\doteq -2.581\,14$

EXERCISE 5a

1. a. $\log_3 x+\log_3 6=\log_3 5$

$x>0$

$\log_3(x)(6)=\log_3 5$

$6x=5$

$x=\frac{5}{6}$

b. $4\log x=\log\frac{1}{16}$, $x>0$

$\log x^4=\log\frac{1}{16}$

$x^4=\frac{1}{16}$

$x=\pm\frac{1}{2}$

Since $x>0$, the solution is $\frac{1}{2}$.

c. $\log 6-\log x=\log 3x+\log 2x$

$\log\frac{6}{x}=\log(3x)(2x)$ and $x>0$

$\frac{6}{x}=6x^2$

$6=6x^3$

$x^3=1$

$x=1$

d. $\log_5 2x-\log_5 7=1$

$\log_5\frac{2x}{7}=1$

$\frac{2x}{7}=5^1$

$2x=35$

$x=17.5$

e. $\log_2 2x + \log_2 (x-1) = 2$

$\log_2 (2x)(x-1) = \log_2 4$

$(2x)(x-1) = 4$

$2x^2 - 2x = 4$

$2x^2 - 2x - 4 = 0$

$2(x^2 - x - 2) = 0$

$2(x+1)(x-2) = 0$

$x = -1$ or $x = 2$

Since $x > 0$, the solution is 2.

f. $2x - 1 = \log_4 8 + \log_4 4 - \log_4 0.5$

$2x - 1 = \log_4 \left(8 \times \frac{4}{0.5}\right)$

$2x - 1 = \log_4 64$

$2x - 1 = 3$

$2x = 4$

$x = 2$

g. $\log \frac{2x}{5} - \log 8 = 2$

$\log \frac{\frac{2x}{5}}{8} = 2$ and $x > 0$

$\log \frac{x}{20} = 2$

$10^2 = \frac{x}{20}$

$x = 100 \times 20$

$x = 2000$

h. $\frac{1}{4}\log x - \frac{1}{2}\log 4x + 2 = 2\log 5 + 2\log 2$

$\log x^{\frac{1}{4}} - \log(4x)^{\frac{1}{2}} + 2 = \log 5^2 + \log 2^2$ and $x > 0$

$\log\left(\frac{x^{\frac{1}{4}}}{2x^{\frac{1}{2}}}\right) + 2 = \log\left(5^2 \times 2^2\right)$

$\log \frac{x^{-\frac{1}{4}}}{2} = \log 100 - 2$

$\log \frac{x^{-\frac{1}{4}}}{2} = 2 - 2$

$\log \frac{x^{-\frac{1}{4}}}{2} = 0$

$\frac{x^{-\frac{1}{4}}}{2} = 10^0$

$\frac{x^{-\frac{1}{4}}}{2} = 1$

$x^{-\frac{1}{4}} = 2$

$\left(x^{-\frac{1}{4}}\right)^{-4} = 2^{-4}$

$x = \frac{1}{16}$

EXERCISE 5b

2. a. $4^x = 9$

$\log 4^x = \log 9$

$x \log 4 = \log 9$

$x = \frac{\log 9}{\log 4}$

$x \doteq 1.585$

b. $4^{\frac{x}{4}} = 23$

$\log 4^{\frac{x}{4}} = \log 23$

$\frac{x}{4}\log 4 = \log 23$

$x = \frac{4\log 23}{\log 4}$

$x \doteq 9.047$

c. $3^{2x} = 4^{5x-2}$

$\log 3^{2x} = \log 4^{5x-2}$

$2x \log 3 = (5x-2)\log 4$

$2x \log 3 = 5x \log 4 - 2 \log 4$

$2x \log 3 - 5x \log 4 = -2 \log 4$

$x(2 \log 3 - 5 \log 4) = -2 \log 4$

$$x = \frac{-2 \log 4}{2 \log 3 - 5 \log 4}$$

$x \doteq 0.586$

3. a. $\log_4 9 = \dfrac{\log 9}{\log 4}$

$\doteq 1.585$

b. $\log_{\frac{1}{5}} 7 = \dfrac{\log 7}{\log \frac{1}{5}}$

$\doteq -1.209$

c. $\dfrac{1}{\log_3 8} = \dfrac{1}{\frac{\log 8}{\log 3}}$

$\doteq 0.528$

EXERCISE 5c

4. a. $A = P(1+i)^n$

$2100 = 700(1+0.05)^n$

$3 = 1.05^n$

$\log 3 = \log 1.05^n$

$\log 3 = n \log (1.05)$

$$n = \frac{\log 3}{\log 1.05}$$

$n \doteq 22.52$ (This is the number of six-month interest periods.)

It will take approximately 11.3 years for the investment to triple.

b. Substitute $M_0 = 40$, $M = 4$ and $h = 28$ into the formula $M = M_0 \left(\frac{1}{2}\right)^{\frac{t}{h}}$.

$4 = 40\left(\frac{1}{2}\right)^{\frac{t}{28}}$

$\frac{1}{10} = \left(\frac{1}{2}\right)^{\frac{t}{28}}$

$\log \frac{1}{10} = \log \left(\frac{1}{2}\right)^{\frac{t}{28}}$

$\log \frac{1}{10} = \frac{t}{28} \log \frac{1}{2}$

$$t = -1 \times \frac{28}{\log \frac{1}{2}}$$

$t \doteq 93.01$

It would take about 93 years for the substance to decay to 4 g.

c. The generic formula $y = ca^x$ can be used with all exponential problems. Substitute $y = 8.03 \times 10^7$, $c = 7.86 \times 10^7$ and $x = 10$ into the formula, and solve for a.

$8.03 \times 10^7 = (7.86 \times 10^7)a^{10}$

$1.021\,63 \doteq a^{10}$

$a \doteq \sqrt[10]{1.021\,63}$

$a \doteq 1.002\,14$

Now substitute $x = 2000 - 1970 = 30$, $a = 1.002\,14$ and $c = 7.86 \times 10^7$ into $y = ca^x$.

$y = 7.86 \times 10^7 (1.002\,14)^{30}$

$= 8.38 \times 10^7$

The population in the year 2000 will be approximately 8.38×10^7.

.I97-402c2 v6

MATH FACTOR

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Lesson 6: Describing Real-World Phenomena Using Logarithms

Student Workbook

Produced by

THE EDUCATION STATION ACCESS

for

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This guide accompanies **Describing Real-World Phenomena Using Logarithms** from the Math Factor series.

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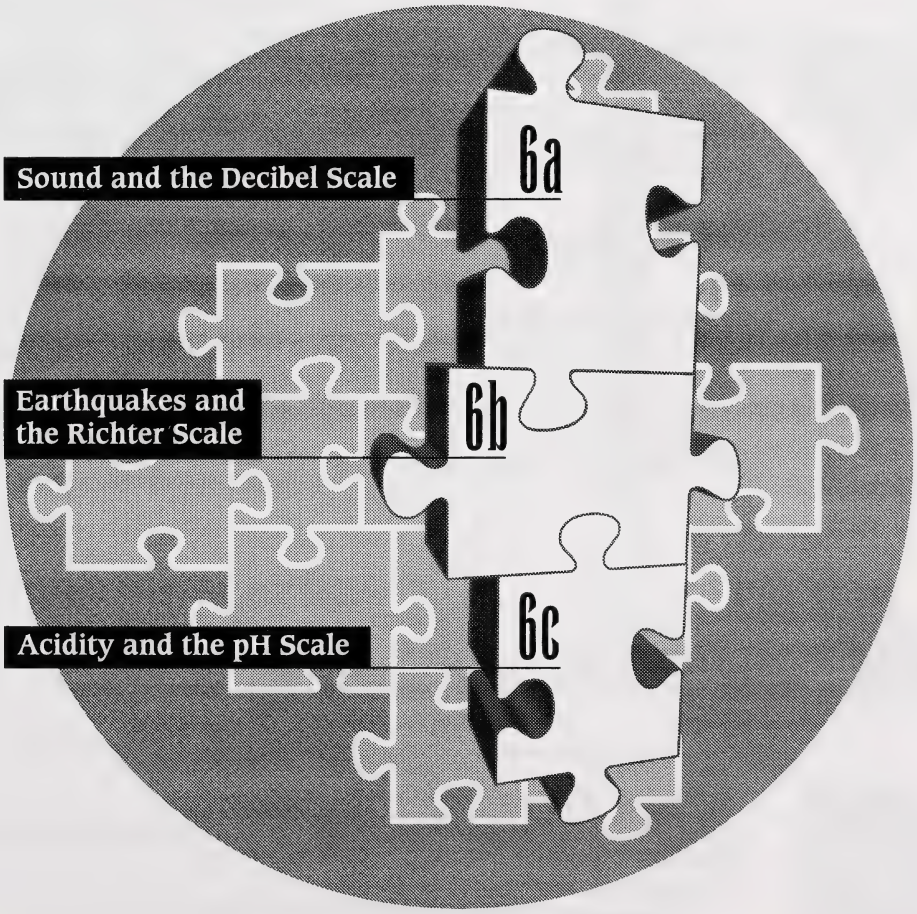
Lesson 6: DESCRIBING REAL-WORLD PHENOMENA USING LOGARITHMS

LEARNER EXPECTATIONS

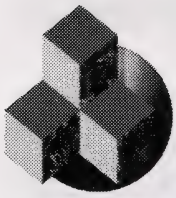
By the end of Lesson 6 you should be able to

- recognize situations that have logarithmic characteristics
- solve problems that exhibit logarithmic properties by developing and solving logarithmic equations

This lesson has three parts:



EXPLANATION OF ICONS



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Video:

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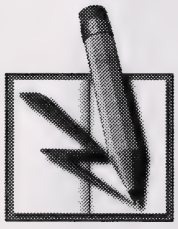

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Sound and the Decibel Scale

Watch the Introduction and Video Segment 6a.

The range of sounds detectable by the human ear is amazing. A person operating a chain saw can create sounds that are ten million times louder than a whisper. The loudness of sounds is measured in **decibels (dB)**. The decibel scale is a logarithmic scale.

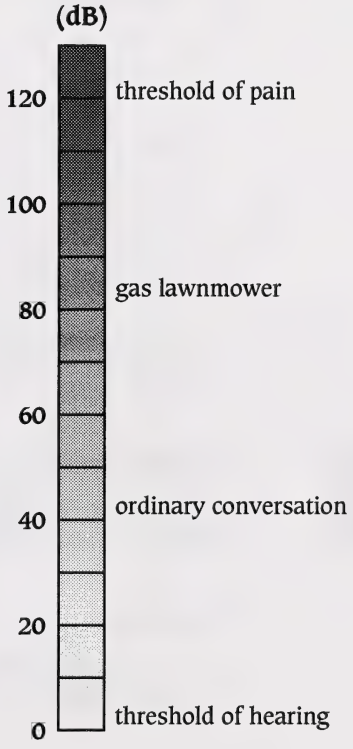
Every increase of 10 dB corresponds to a tenfold increase in loudness.

If you have to raise your voice to be heard, the background noise level would measure approximately 85 dB. **Continuous exposure to noise levels greater than 85 dB can lead to hearing loss.**

A sound that is just detectable (the threshold of hearing) is assigned an intensity of 1. The intensities of other sounds are compared to this reference intensity. A sound with an intensity of 1000 has a decibel measure of 30. It is 1000 times as intense as the threshold of hearing that has a decibel measure of 0.

Many problems involving sound can be solved using the formula $s_1 - s_2 = 10 \log \frac{L_1}{L_2}$, where L_1 and L_2 are the intensities of two sounds and s_1 and s_2 are their respective loudness measurements in decibels.

The Decibel Scale



Example 1

The noise on a busy street is measured at 80 dB. The sound of an ordinary conversation is measured at 40 dB. How many times louder is the noise on the busy street compared to the sound of the ordinary conversation?

Solution

Let $s_1 = 80\ dB$ and $s_2 = 40\ dB$.

$$80 - 40 = 10 \log \frac{L_1}{L_2}$$

$$= 10 \log \frac{L_1}{L_2}$$

$$s_1 - s_2 = 40$$

$$40 = 10 \log \frac{L_1}{L_2}$$

$$4 = \log \frac{L_1}{L_2}$$

$$10^4 = \frac{L_1}{L_2}$$

$$L_1 = 10^4 L_2$$

$$L_1 = 10\,000 L_2$$

The street noise is 10 000 times more intense (or louder) than the sound of an ordinary conversation.

Additional Examples
Mathematics 12: p. 312
Mathematics: Principles and Process 12: p. 114
Holtmath 12: p. 142 (Example 2)

Exercises

1. How many times louder is a sound at the 50 dB level compared to a sound at the 25 dB level?

2. Before being repaired, a gas lawnmower with a defective muffler had a loudness level of 108 dB. A new muffler was installed, reducing the loudness of the mower by a factor of 25. Determine the loudness level of the repaired lawnmower.

3. A sound has a reading of 19.3 dB. Determine its intensity.

EXERCISE 6a SOLUTIONS are at the back of this workbook.

Additional Exercises
Mathematics 12: pp. 312, 313 (Questions 1-8)
Mathematics: Principles and Process 12: p. 114 (Questions 7-11)
Holtmath 12: p. 143 (Question 7)

Earthquakes and the Richter Scale

challenge yourself

Given the measure of an earthquake to be 3.4 on the Richter scale, find the measurement of an earthquake that would be twice as strong.

Watch Video Segment 6b.

The **Richter scale** is used to determine the intensity of an earthquake compared to a predetermined standard that has a measure of 0 on the scale. The Richter scale is a logarithmic scale.

Major Earthquakes of the Eighties Richter Scale

Southeast Mexico (5600 deaths)

Honshu, Japan (104 deaths), Chile (177 deaths)

California (63 deaths)

Southern Italy (1000 deaths), Turkey (2700 deaths) Afghanistan (25 000 deaths)

Because the difference in intensities can be enormous, taking the logarithm base 10 of these intensities gives a scale that is workable.

Richter scale numbers have no units.

An increase of one on the Richter scale means that the intensity of an earthquake is 10 times greater, but the total energy released by the earthquake is more complicated. It increases by about 30 times.

The formula $m = \log \frac{I_1}{I_0}$, where I_1 is the intensity of the earthquake being measured and I_0, the intensity of a reference earthquake, can be used to determine the measured earthquake's Richter scale reading, m.

Example

On the Richter scale, Earthquake A has a magnitude of 7.8 and Earthquake B has a magnitude of 5.4. How many times more intense is Earthquake A compared to Earthquake B?

Solution

Use $m = \log \frac{I_1}{I_0}$.

If $m_1 = 7.8$, then $7.8 = \log \frac{I_1}{I_0}$.

$$\frac{I_1}{I_0} = 10^{7.8}$$

$$I_1 = 10^{7.8} I_0 \quad \text{(Equation 1)}$$

If $m_2 = 5.4$, then $5.4 = \log \frac{I_2}{I_0}$.

$$\frac{I_2}{I_0} = 10^{5.4}$$

$$I_2 = 10^{5.4} I_0 \quad \text{(Equation 2)}$$

$$\frac{I_1}{I_2} = \frac{10^{7.8} I_0}{10^{5.4} I_0} \quad \text{(Equation 1 divided by Equation 2.)}$$

$$\frac{I_1}{I_2} = 10^{7.8-5.4}$$

$$= 10^{2.4}$$

$$\doteq 251$$

Earthquake A is approximately 251 times more intense than Earthquake B.

Note: Problems using the Richter scale can be solved using a formula similar to the one dealing with sound.

$$m_1 - m_2 = \log \frac{I_1}{I_2}$$

$$7.8 - 5.4 = \log \frac{I_1}{I_2}$$

$$2.4 = \log \frac{I_1}{I_2}$$

$$\therefore 10^{2.4} = \frac{I_1}{I_2}$$

$$\frac{I_1}{I_2} \doteq 251$$

Additional Examples
Mathematics 12: p. 358

Exercises

4. Determine the Richter scale reading of an earthquake that is 25 times more intense than an earthquake measuring 0 on the Richter scale.

5. Compare the intensity of an earthquake with a Richter scale reading of 6.5 to the intensity of an earthquake measuring 0 on the Richter scale.

6. An earthquake in Northeast Italy in 1976 with a Richter scale reading of 6.5 caused over 1000 deaths. In 1978, an earthquake in Iran with a Richter scale reading of 7.4 caused over 20 000 deaths. How many times more intense was the earthquake in Iran compared to the one in Italy?

7. Determine the Richter scale reading of an earthquake that is 50 times as intense as an earthquake with a Richter scale reading of 3.6.

EXERCISE 6b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 360 (Questions 4-6)
Mathematics: Principles and Process 12: p. 140 (Question 10)
Holtmath 12: p. 143 (Question 8)

Acidity and the pH Scale

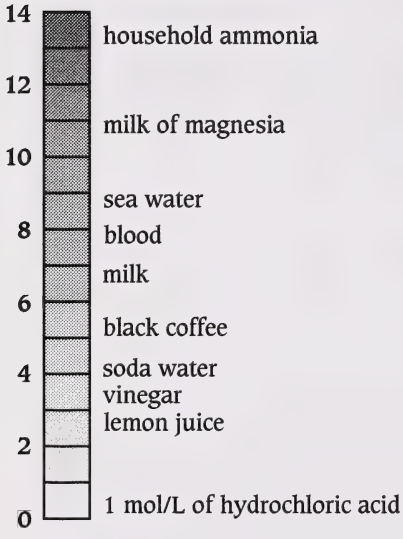
Challenge yourself

Find the pH of a soft drink that has a hydrogen ion concentration of 8.1×10^{-4} mol/L.

Watch Video Segment 6c.

Acidity is a measure of the hydrogen ion concentration of a solution. Because such a large range of values is possible, a Danish chemist by the name of Soren Sorenson came up with the idea of using logarithms to put this range into a manageable scale.

pH of Various Solutions



The pH of a solution is the negative logarithm base 10 of the hydrogen ion concentration.

$$\text{pH} = -\log\left[\text{H}^+\right]$$

The lower the pH of a solution, the more acidic that solution is.

The *p* stands for *potenz*, which is Danish for power or exponent.

At 25°C and **pH = 7**, a solution is said to be **neutral**.

If **pH > 7**, the solution is said to be **alkaline**.

If **pH < 7**, the solution is said to be **acidic**.

Example

Jen doubled the acidity of a solution of hydrochloric acid. If the solution had an initial pH of 2.5, determine the solution's final pH.

Solution

Initial Acid

$$2.5 = -\log\left[H^+\right]$$

$$\left[H^+\right] = 10^{-2.5}$$

Final Acid

$$x = -\log\left[H^+\right]$$

$$\left[H^+\right] = 10^{-x}$$

Since $\dfrac{\left[H^+\right] \text{ of Final Acid}}{\left[H^+\right] \text{ of Initial Acid}} = 2$, then $2 = \dfrac{10^{-x}}{10^{-2.5}}$.

$$2 = \frac{10^{-x}}{10^{-2.5}}$$

$$2 \times 10^{-2.5} = 10^{-x}$$

$$\log\left(2 \times 10^{-2.5}\right) = \log\left(10^{-x}\right)$$

$$\log\left(2 \times 10^{-2.5}\right) = -x \log 10$$

$$-\log\left(2 \times 10^{-2.5}\right) = x \log(10)$$

$$\frac{-\log\left(2 \times 10^{-2.5}\right)}{\log 10} = x$$

$$x = -\frac{\left(\log 2 + \log 10^{-2.5}\right)}{\log 10}$$

$$x \doteq 2.20$$

The pH of the final solution would be approximately 2.20.

Additional Examples
Mathematics 12: p. 359

Exercises

8. Determine the hydrogen ion concentration of each of the following solutions.

   a. a sample of soda or soft-drink with a pH of 3.61

b. a sample of seawater with a pH of 8.64

c. a sample of human saliva with a pH of 6.65

9. The pH of a liquid can be measured with a simple-to-use pH meter equipped with electrodes. Using a pH meter, Jurgen obtained a pH reading of 2.9 for Solution A. If Solution B is 30 times less acidic than Solution A, determine the pH reading Jurgen would obtain for Solution B.

EXERCISE 6c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 361 (Questions 7-9)
Holtmath 12: p. 143 (Question 5)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points you have mastered. You should now realize that there are many applications for the mathematics of logarithms. The skills you have acquired and practised in this and previous lessons will be reviewed in **Lesson 7: Review and Problem Solving.**

Solutions

EXERCISE 6a

1. $s_1 = 50$ and $s_2 = 25$

$$s_1 - s_2 = 10 \log \frac{L_1}{L_2}$$
$$50 - 25 = 10 \log \frac{L_1}{L_2}$$
$$25 = 10 \log \frac{L_1}{L_2}$$
$$2.5 = \log \frac{L_1}{L_2}$$
$$10^{2.5} = \frac{L_1}{L_2}$$
$$L_1 = 10^{2.5} L_2$$
$$L_1 \doteq 316 L_2$$

The sound at 50 dB is about 316 times louder than the sound at 25 dB.

2. $s_1 = 108$ and $\frac{L_1}{L_2} = 25$

$$s_1 - s_2 = 10 \log \frac{L_1}{L_2}$$
$$108 - s_2 = 10 \log 25$$
$$s_2 = 108 - 10 \log 25$$
$$s_2 \doteq 94$$

The repaired lawnmower operates at a loudness level of approximately 94 dB.

3. Compare the sound to the threshold of hearing which has an intensity of 1 and a decibel reading of 0.

$s_1 = 19.3$, $s_2 = 0$ and $L_2 = 1$

$$s_1 - s_2 = 10 \log \frac{L_1}{L_2}$$
$$19.3 - 0 = 10 \log \frac{L_1}{1}$$
$$1.93 = \log L_1$$
$$L_1 = 10^{1.93}$$
$$L_1 \doteq 85$$

The intensity of the sound is approximately 85 dB.

EXERCISE 6b

4. $I_1 = 25 I_0$

$$m = \log \frac{I_1}{I_0}$$
$$= \log \frac{25 I_0}{I_0}$$
$$= \log 25$$
$$\doteq 1.41$$

The Richter scale reading of an earthquake that is 25 times more intense than an earthquake measuring 0 on the Richter scale is approximately 1.41.

5. $I_0 = 1$ and $m = 6.5$

$$m = \log \frac{I_1}{I_0}$$
$$6.5 = \log I_1$$
$$10^{6.5} = I_1$$
$$I_1 \doteq 3.16 \times 10^6$$

An earthquake with a measure of 6.5 on the Richter scale is approximately 3.16×10^6 times more intense than the earthquake measuring 0 on the Richter scale.

6. Use $m = \log \frac{I_1}{I_0}$.

$m_1 = 7.4$ and $m_2 = 6.5$

$7.4 = \log \frac{I_1}{I_0}$ and $6.5 = \log \frac{I_2}{I_0}$

$10^{7.4} = \frac{I_1}{I_0}$ $\qquad$ $10^{6.5} = \frac{I_2}{I_0}$

$\therefore \frac{10^{7.4}}{10^{6.5}} = \frac{I_1}{I_2}$

$\frac{I_1}{I_2} = 10^{0.9}$

$\frac{I_1}{I_2} \doteq 8$

The earthquake in Iran was approximately eight times more intense than the earthquake in Italy.

7. Use $m = \log \frac{I_1}{I_0}$.

$m_1 = 3.6$ and $\frac{I_1}{I_2} = 50$

$m_1 = \log \frac{I_1}{I_0}$ and $3.6 = \log \frac{I_2}{I_0}$

$10^{m_1} = \frac{I_1}{I_0}$ $\qquad$ $10^{3.6} = \frac{I_2}{I_0}$

$\frac{10^{m_1}}{10^{3.6}} = \frac{I_1}{I_2}$

$\frac{10^{m_1}}{10^{3.6}} = 50$

$10^{m_1} = \left(10^{3.6}\right)(50)$

$\log 10^{m_1} = \log\left(10^{3.6}\right)(50)$

$\log 10^{m_1} = \log 10^{3.6} + \log 50$

$m_1 = 3.6 + \log 50$

$m_1 \doteq 5.3$

The Richter scale reading is approximately 5.3.

EXERCISE 6c

8. a. $\text{pH} = -\log\left[\text{H}^+\right]$

$3.61 = -\log\left[\text{H}^+\right]$

$10^{-3.61} = \left[\text{H}^+\right]$

$\left[\text{H}^+\right] \doteq 2.45 \times 10^{-4}$

b. $8.64 = -\log\left[\text{H}^+\right]$

$10^{-8.64} = \left[\text{H}^+\right]$

$\left[\text{H}^+\right] \doteq 2 \times 10^{-9}$

c. $6.65 = -\log\left[\text{H}^+\right]$

$10^{-6.65} = \left[\text{H}^+\right]$

$\left[\text{H}^+\right] \doteq 2.23 \times 10^{-7}$

9. Use $\text{pH} = -\log\left[\text{H}^+\right]$.

Solution A $\qquad$ **Solution B**

$2.9 = -\log\left[\text{H}^+\right]$ $\qquad$ $x = -\log\left[\text{H}^+\right]$

$10^{-2.9} = \left[\text{H}^+\right]$ $\qquad$ $10^{-x} = \left[\text{H}^+\right]$

$\frac{\left[\text{H}^+\right] \text{ Solution A}}{\left[\text{H}^+\right] \text{ Solution B}} = \frac{30}{1}$

$\frac{10^{-2.9}}{10^{-x}} = 30$

$\frac{10^{-2.9}}{30} = 10^{-x}$

$\log\left(\frac{10^{-2.9}}{30}\right) = \log 10^{-x}$

$\log\left(\frac{10^{-2.9}}{30}\right) = -x$

$x = -\log\left(\frac{10^{-2.9}}{30}\right)$

$x \doteq 4.4$

Jurgen would obtain a pH reading of approximately 4.4.

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MATH FACTOR

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Lesson 7: Review and Problem Solving

Student Workbook

Produced by

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This guide accompanies **Review and Problem Solving** from the Math Factor series.

Writer: Gordon Mallett
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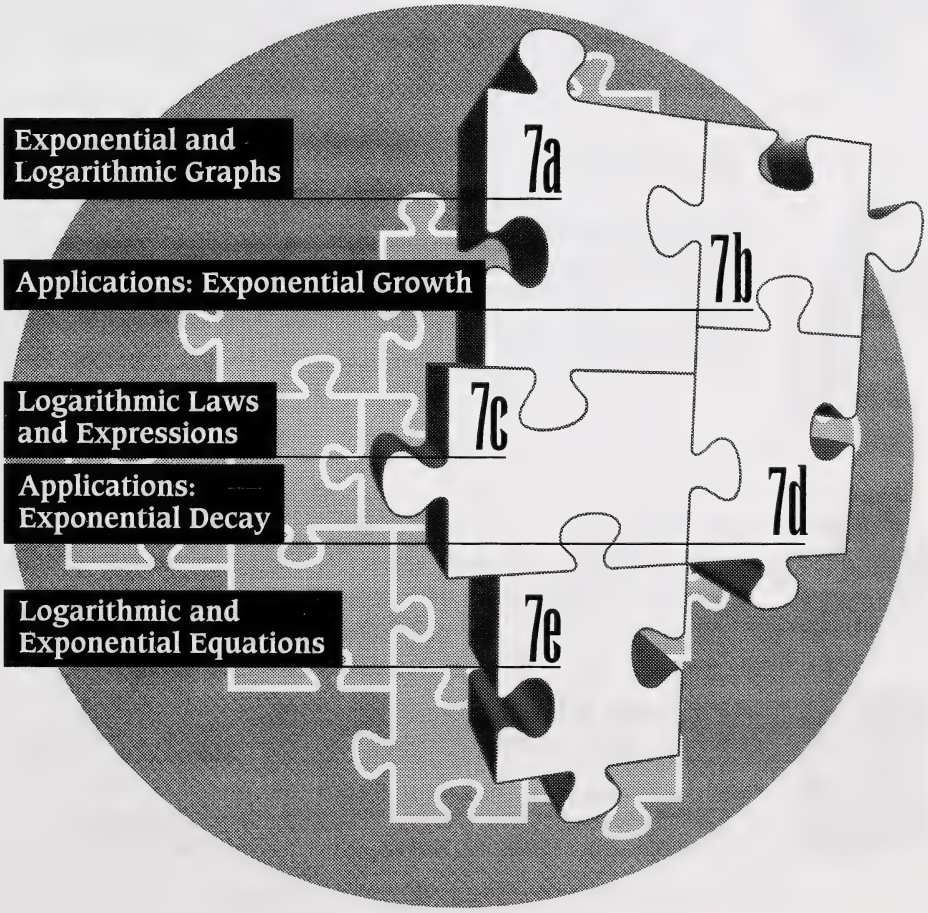
Lesson 7: REVIEW AND PROBLEM SOLVING

LEARNER EXPECTATIONS

By the end of Lesson 7 you should be able to

- investigate the graphs of exponential and logarithmic functions by sketching, using a calculator or using a computer graphing program
- apply the laws and properties of logarithms to simplify or evaluate expressions
- solve practical problems involving exponential growth and decay
- solve exponential and logarithmic equations

This lesson has five parts:



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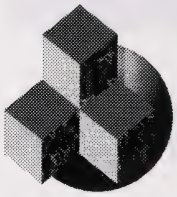

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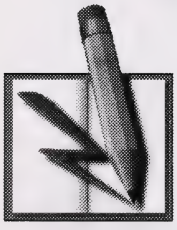

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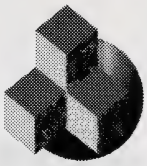


Building Blocks

Do you remember how to apply the laws of exponents? Before watching this program, refresh your memory by studying the following examples and doing the exercises.

Simplifying exponential expressions often involves changing factors (or terms) to the same base. Answers are sometimes left in exponential form.

Example 1

Simplify $\dfrac{\left(2^{n+3} \times 4^{n+2}\right)}{8^n}$.

Solution

$$\frac{\left(2^{n+3} \times 4^{n+2}\right)}{8^n} = \frac{\left(2^{n+3} \times \left(2^2\right)^{n+2}\right)}{\left(2^3\right)^n}$$

$$= \frac{\left(2^{n+3} \times 2^{2n+4}\right)}{2^{3n}}$$

$$= \frac{2^{3n+7}}{2^{3n}}$$

$$= 2^7$$

Solving exponential equations can sometimes be done by expressing both sides of the equation to the same base.

Example 2

Solve $8^{-2n} = \dfrac{1}{16}$.

Solution

$$8^{-2n} = \frac{1}{16}$$

$$\left(2^3\right)^{-2n} = \frac{1}{2^4}$$

$$2^{-6n} = 2^{-4}$$

$$\therefore -6n = -4$$

$$n = \frac{-4}{-6}$$

$$= \frac{2}{3}$$

Definitions

- $a^0 = 1 \qquad (a \neq 0)$
- $a^{-n} = \dfrac{1}{a^n} \qquad (a \neq 0)$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a^m}\right) \qquad (a > 0)$

Laws of Exponents

Note: m and n are any real numbers.

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0)$
- $\left(a^m\right)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Exercises

1. Simplify $\dfrac{\left(3^{2n} \times 3^{-n}\right)}{\left(3^{n} + 3^{n}\right)}$.

2. Solve $3^{1-n} = \dfrac{1}{9^{-2n}}$.

Note: If you had difficulty with these questions, see Video 3 of *Exponential and Logarithmic Functions*.

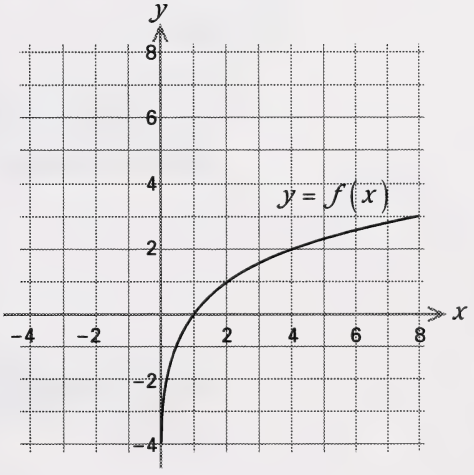
Building Blocks Solutions are at the back of this workbook.

Exponential and Logarithmic Graphs

Watch the Introduction and Video Segment 7a.

Example

The graph of the function $y = f(x)$ where $f(x) = \log_2 x$ is given. Use the graph to approximate (to one decimal place) the value of $\log_2 6$. Then state (in exponential form) the inverse of the above function and sketch the graph of the inverse of $y = f(x)$.



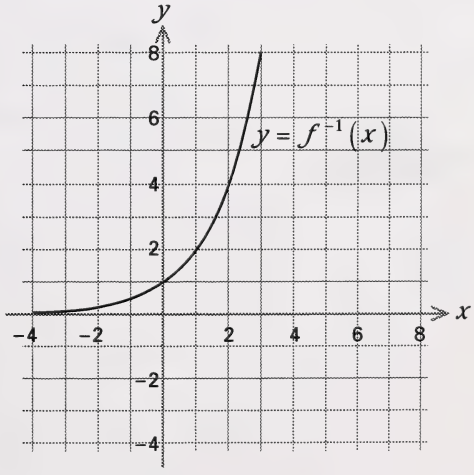
Solution

Locate 6 on the x-axis. Move up until you intersect the graph. According to the graph, the approximate value is 2.6.

The inverse of $y = f(x)$ is as follows:

$x = \log_2 y$

$y = 2^x$



Note: If you had difficulty with this example, see Video 3 of *Exponential and Logarithmic Functions*.

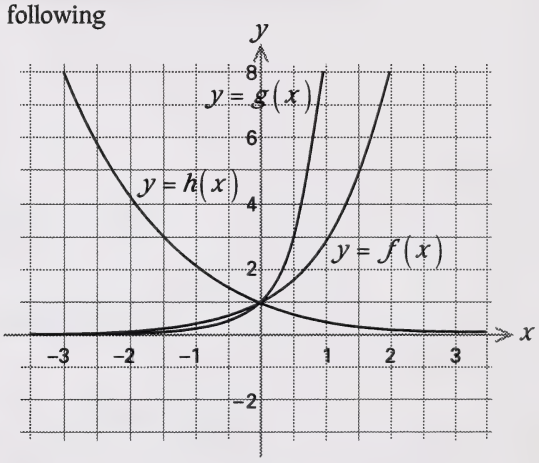
Exercises

1. Find the inverse of each function, stating the answer in exponential and logarithmic form.

   a. $f(x) = 0.4^x$

   b. $g(x) = \log_{\frac{1}{2}} x$

2. Study the graphs of the exponential functions $y = f(x)$, $y = g(x)$ and $y = h(x)$, where $f(x) = 3^x$, $g(x) = 10^x$ and $h(x) = 0.5^x$. Then answer the following questions.



   a. Through what common point do all the graphs pass? What name could be used to describe this point?

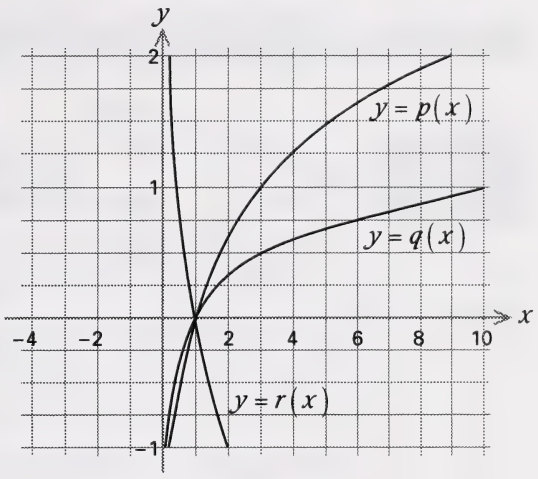
   b. Describe what happens to the graph of exponential functions as the value of the base increases.

EXCELLENCE

   c. Use the graphs to solve the equation $g(x) - h(x) = 0$.

EXCELLENCE

3. Study the graphs of the logarithmic functions $y = p(x)$, $y = q(x)$ and $y = r(x)$, where $p(x) = \log_3 x$, $q(x) = \log x$ and $r(x) = \log_{0.5} x$.



   a. The general form of the logarithmic function is $\log_b x$. Use these graphs to describe when $\log_b x > 0$.

   b. Match each function with its inverse function in question 2.

EXERCISE 7a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 340 (Questions 1-6)
Mathematics: Principles and Process 12: p. 111 (Questions 1-3)
Holtmath 12: p. 127 (Question 4), p. 131 (Question 9)

Applications: Exponential Growth

Watch Video Segment 7b.

There are many real-world phenomena, such as world population, that involve exponential growth. The functions that represent these situations have defining equations of the form $f(x) = ca^x$, where c and a are constants and $a > 0$.

Example

The number of a certain type of bacteria doubles every four hours. To what size will an initial population of 1000 bacteria grow in 26 h?

Solution

It may be helpful to rewrite the generic formula $y = ca^x$ in the form $A = A_0(2)^n$, where A is the final amount, A_0 is the initial amount and n is the number of times the population doubles.

$A = ?,\ A_0 = 1000$ and $n = \frac{26}{4}$

$= 6.5$

$A = A_0(2)^n$

$= 1000 \times 2^{6.5}$

$\doteq 1000 \times 90.5096$

$\doteq 90\ 510$

The bacteria will grow to about 90 510.

Note: If you had difficulty with this example, see Video 2 of *Exponential and Logarithmic Functions*.

Additional Practice
Mathematics 12: p. 300, p. 302 (Questions 1, 2, 9, 10)
Mathematics: Principles and Process 12: p. 136 (Question 2)
Holtmath 12: p. 143 (Question 4)

Logarithmic Laws and Expressions

Find the value of $\log_{\sqrt{3}} 81^{\frac{2}{5}}$.

Watch Video Segment 7c.

When simplifying or manipulating logarithmic expressions, do the following:

- Apply, as required, one or more of the logarithmic laws.

$$\log_a (xy) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

- Change, as required, from logarithmic form to exponential form.
- Apply, as required, the exponent laws and basic procedures required to simplify arithmetic and algebraic expressions.

Example 1

Express $1000 = 10^3$ in logarithmic form.

Solution

$\log_{10} 1000 = 3$

Example 2

Express $2^{-4} = \frac{1}{16}$ in logarithmic form.

Solution

$\log_2 \frac{1}{16} = -4$

Example 3

Express $\log_{10} 0.001 = -3$ in exponential form.

Solution

$10^{-3} = 0.001$

Example 4

Express $4 = \log_5 625$ in exponential form.

Solution

$5^4 = 625$

Example 5

Find the value of $\log_4 (\log_3 81)$.

Solution

$$\begin{aligned}\log_3 81 &= \log_3 3^4 \\ &= 4 \log_3 3 \\ &= 4 \times 1 \\ &= 4\end{aligned}$$

$$\begin{aligned}\therefore \log_4 (\log_3 81) &= \log_4 4 \\ &= 1\end{aligned}$$

Example 6

Determine an expression for $\log_a a^x$.

Solution

$$\begin{aligned}\log_a a^x &= x \log_a a \\ &= x \times 1 \\ &= x\end{aligned}$$

Note: If you had difficulty with Examples 1 to 6, see Video 4 of *Exponential and Logarithmic Functions*.

Example 7

Use the laws of logarithms to expand $\log_3 \frac{\sqrt{ab}}{c}$.

Solution

$$\log_3 \frac{\sqrt{ab}}{c} = \log_3 \sqrt{ab} - \log_3 c$$

$$= \log_3 (ab)^{\frac{1}{2}} - \log_3 c \qquad \text{(logarithmic law for division)}$$

$$= \frac{1}{2}\log_3 (ab) - \log_3 c \qquad \text{(logarithmic law for powers)}$$

$$= \frac{1}{2}(\log_3 a + \log_3 b) - \log_3 c \qquad \text{(logarithmic law for multiplication)}$$

$$= \frac{1}{2}\log_3 a + \frac{1}{2}\log_3 b - \log_3 c$$

Example 8

Write $\log_4 8 + 2\log_4 3 - \log_4 12$ as a single logarithmic expression.

Solution

$$\log_4 8 + 2\log_4 3 - \log_4 12 = \log_4 8 + \log_4 3^2 - \log_4 12$$

$$= \log_4 \left(8 \times \frac{3^2}{12}\right)$$

$$= \log_4 6$$

Additional Examples
Mathematics 12: pp. 324-327, 342, 343, 346-348
Mathematics: Principles and Process 12: p. 112 (Examples 1, 2), pp. 119, 121, 123

Exercises

4. Find the value of $\log_4 \left(\log_5 5^2\right)$.

5. Find the value of $\log_2 64^{\frac{3}{5}}$.

6. Use the laws of logarithms to determine the value of $\log_4 48 + \log_4 \frac{2}{3} - \log_4 8$.

7. Write $5\log_5 m + \log_5 n - \log_5 m^2$ as a single logarithmic expression.

8. If $v = \pi r^2 h$, write $\log r$ as a sum or difference of logs.

EXERCISE 7c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 327 (Questions 1, 9, 11), p. 343 (Questions 1, 2, 6, 7), p. 349 (Question 8)
Mathematics: Principles and Process 12: p. 113 (Questions 1-3), p. 120 (Question 4), p. 122 (Questions 3-5), p. 124 (Questions 3, 6)
Holtmath 12: p. 146 (Questions 3, 11)

Applications: Exponential Decay

Challenge yourself

A filter loses one-half percent of its effectiveness each day. What percentage of its original effectiveness does the filter have after 100 days of use?

MATH FACTOR

Watch Video Segment 7d.

Functions that represent exponential decay have defining equations of the form $f(x) = ca^x$, where $0 < a < 1$. Logarithms are often used to solve these as well as other types of real-life applications.

Example

The half-life of strontium-90 is 28 years. How long will it take before only 10% of a sample remains?

Solution

Let t be the required length of time (in years) and A_0 be the original amount of strontium-90. Thus 10% A_0 is the amount remaining after the required length of time.

After each half-life, the amount of strontium-90 remaining is halved. So, the time until only 10% remains is given by the following:

$$10\%\, A_0 = A_0\left(\frac{1}{2}\right)^{\frac{t}{28}}$$

$$0.10\, A_0 = A_0(0.5)^{\frac{t}{28}}$$

$$0.10 = 0.5^{\frac{t}{28}}$$

$$\log 0.10 = \log 0.5^{\frac{t}{28}}$$

$$\log 0.10 = \frac{t}{28}\log 0.5$$

$$\frac{t}{28} = \frac{\log 0.10}{\log 0.5}$$

$$t = 28 \times \frac{\log 0.10}{\log 0.5}$$

$$t \doteq 93$$

It will take about 93 years for 10% of a strontium-90 sample to remain.

If you had difficulty with this example, see Video 5 of *Exponential and Logarithmic Functions*.

Additional Examples
Mathematics 12: pp. 301, 334-336, 351-353, 357-359
Mathematics: Principles and Process 12: p. 136 (Example 6)

Exercises

9. When a diver descends below the water surface, the intensity of light is reduced by 2.5% for every metre the diver descends. The equation giving the depth at which the light intensity is 50% of that at the surface is $500 = 1000(0.975)^d$. Solve the equation for d (to the nearest tenth).

10. If the half-life of a radioactive element is 20 weeks, then how much of a 250 g sample will remain after 30 weeks (correct to the nearest gram)?

EXERCISE 7d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 303 (Questions 5, 11), p. 337 (Question 4), p. 355 (Question 16), p. 360 (Questions 4, 6)
Holtmath 12: p. 145 (Questions 16, 18)

Logarithmic and Exponential Equations

Solve the equation $\log_6(x-5)+\log_6(x-6)=\log_6$.

Watch Video Segment 7c.

When solving exponential and logarithmic equations, do the following:

- Recall that exponential and logarithmic forms are interchangeable.
- Apply the laws of exponents and laws of logarithms as required.
- Keep the equations simplified by arithmetic and algebraic reductions.

Example 1

Solve $2\log x = \log 32 + \log 2$. Verify your answer.

Solution

$2\log x = \log 32 + \log 2$

$2\log x = \log(32 \times 2)$ (Multiplication Law of Logarithms)

$\log x^2 = \log 64$ (Power Law of Logarithms)

$x^2 = 64$

$x = \pm 8$

Verification:

The left side is not defined for $x = -8$. Hence, -8 is an extraneous root.

L.S.	R.S.
$2\log 8$	$\log 32 + \log 2$
$= \log 8^2$	$= \log(32 \times 2)$
$= \log 64$	$= \log 64$

L.S. = R.S.

Therefore, $x = 8$ is the solution.

Example 2

If $\log_x \frac{4}{9} = -2$, find x.

Solution

$$\log_x \frac{4}{9} = -2$$

$$x^{-2} = \frac{4}{9}$$

$$\frac{1}{x^2} = \frac{4}{9}$$

$$4x^2 = 9$$

$$x^2 = \frac{9}{4}$$

$$x = \pm\sqrt{\frac{9}{4}}$$

$$= \pm\frac{3}{2}$$

Bases of logarithms are never negative; therefore, $x = \frac{3}{2}$ is the solution.

Additional Examples
Mathematics 12: p. 348 (Example 5)
Mathematics: Principles and Process 12: p. 127
Holtmath 12: p. 140

Exercises

11. If $\log_2(1-x) - \log_2(11+x) = -1$, find $\log_2(5+x)$.

12. If $\log_2 256 = x + y$ and $\log_8 64 = x - y$, find the value of y.

If you had difficulty with these questions, see Video 5 of *Exponential and Logarithmic Functions*.

EXERCISE 7e SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 344 (Questions 8, 9), p. 350 (Question 18)
Mathematics: Principles and Process 12: p. 127 (Questions 1-5)
Holtmath 12: p. 141 (Questions 6, 10)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Go back to the videos and study those parts that you find difficult. Now that you have finished **Exponential and Logarithmic Functions**, practise your skills by working on recent diploma exam questions involving logarithmic and exponential functions.

Solutions

BUILDING BLOCKS

1. a. $\dfrac{\left(3^{2n}\times 3^{-n}\right)}{\left(3^{n}+3^{n}\right)} = \dfrac{3^{2n+(-n)}}{3^{n}(1+1)}$

$= \dfrac{3^{n}}{3^{n}(2)}$

$= \dfrac{1}{2}$

b. $3^{(1-n)} = \dfrac{1}{9^{-2n}}$

$3^{(1-n)} = 9^{2n}$

$3^{(1-n)} = 3^{4n}$

$1-n = 4n$

$1 = 5n$

$n = \dfrac{1}{5}$

EXERCISE 7a

1. a. $f(x) = 0.4^{y}$

$y = \log_{0.4} x$

b. $g(x) = \log_{\frac{1}{2}} y$

$y = \left(\dfrac{1}{2}\right)^{x}$

2. a. The graphs all pass through $(0,1)$. This point is the y-intercept of these graphs.

b. When the base is between 0 and 1, the function is decreasing. When the value of the base is greater than 1, the function is increasing. As the values of the base get larger, the graph rises more rapidly from left to right.

c. $g(x) - h(x) = 0$

$\therefore g(x) = h(x)$

This occurs only at the point $(0,1)$. So, the solution is $x = 0$.

3. a. For $\log_b x$ to be greater than 0, $x > 1$ and $b > 1$ (or $0 < x < 1$ and $0 < b < 1$).

b. $p(x)$ is the inverse of $f(x)$; $q(x)$ is the inverse of $g(x)$;
and $r(x)$ is the inverse of $h(x)$.

EXERCISE 7c

4. $\log_4\left(\log_5 5^2\right) = \log_4\left(2\log_5 5\right)$

$= \log_4(2\times 1)$

$= \log_4 2$

$= \log_4 4^{\frac{1}{2}}$ (2 is rewritten as $4^{\frac{1}{2}}$.)

$= \dfrac{1}{2}\log_4 4$

$= \dfrac{1}{2}$

5. $\log_2 64^{\frac{3}{5}} = \log_2\left(2^6\right)^{\frac{3}{5}}$

$= \log_2 2^{\frac{18}{5}}$

$= \dfrac{18}{5}\log_2 2$

$= \dfrac{18}{5}$

6. $\log_4 48 + \log_4 \frac{2}{3} - \log_4 8 = \log_4 \frac{\left(48 \times \frac{2}{3}\right)}{8}$

$= \log_4 4$

$= 1$

7. $5\log_5 m + \log_5 n - \log_5 m^2 = \log_5 m^5 + \log_5 n - \log_5 m^2$

$= \log_5 \left(m^5 \times \frac{n}{m^2}\right)$

$= \log_5 m^3 n$

8. $v = \pi r^2 h$

$\frac{v}{\pi h} = r^2$

$\left(\frac{v}{\pi h}\right)^{\frac{1}{2}} = r$

$\log r = \log\left(\frac{v}{\pi h}\right)^{\frac{1}{2}}$

$= \frac{1}{2}(\log v - \log \pi h)$

$= \frac{1}{2}(\log v - \log \pi - \log h)$

EXERCISE 7d

9. $1000(0.975)^d = 500$

$(0.975)^d = 0.5$

$\log(0.975)^d = \log 0.5$

$d \log(0.975) = \log 0.5$

$d = \frac{\log 0.5}{\log(0.975)}$

$\doteq 27.4$

The diver descended to a depth of about 27.4 m.

10. $A = 250\left(\frac{1}{2}\right)^{\frac{30}{20}}$

$= 250(0.5)^{1.5}$

$\doteq 88$

About 88 g will remain after 30 weeks.

EXERCISE 7e

11. $\log_2(1-x)-\log_2(11+x)=-1$

$$\log_2\left(\frac{1-x}{11+x}\right)=-1$$

$$\frac{1-x}{11+x}=2^{-1}$$

$$2(1-x)=1(11+x)$$

$$2-2x=11+x$$

$$-3x=9$$

$$x=-3$$

Checking $x=-3$ confirms the solution.

$$\therefore \log_2(5+x)=\log_2[5+(-3)]$$

$$=\log_2 2$$

$$=1$$

12. $\log_2 256=\log_2 2^8$ $\qquad$ $\log_8 64=\log_8 8^2$

$=8$ $\qquad$ $=2$

$\therefore 8=x+y$ $\qquad$ $\therefore 2=x-y$

Solve the system.

$$8=x+y$$

$$2=x-y$$

$$8-2=(x+y)-(x-y)$$

$$6=x+y-x+y$$

$$6=2y$$

$$y=3$$